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TTX Risk Measures Derived from General Survival Theory

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Probabilistic Unified Risk Estimation: General Survival Theory (GST) and TTX Risk Measures

Julian Eggert[1]

Abstract— **Although intelligent Advanced Driving Assistance Systems and Autonomous Driving technologies are becoming ubiquitous, efficient and safe driving in dynamic, narrow or congested environment remains a theoretical and technological challenge. The reason mainly lies in the complexity of handling prediction and uncertainty, which are the fundamental ingredients for risk estimation. The theoretical shortcomings can be seen on the level of mesoscopic risk quantifiers like those based on Time-To-Event (TTX, e.g. TTC, TTB) or spatial safe distances, which mainly reflect heuristic engineering practices or worst case deterministic forecasts of a scenario. In both cases, uncertainties are not explicitly considered and worst-case assumptions significantly reduce driving efficiency, like traffic throughput. However uncertainties are ubiquitous in real driving so that a fundamentally different strategy is required, which seeks for a tradeoff between risk and benefit. In this paper, we introduce a microscopic, probabilistic unified risk estimation framework (PURE), grounded in a first principles probabilistic risk theory, which generalizes to arbitrary risks and situations and which explicitly addresses these uncertainties. To underline the generality of the approach, we show how empirical risk measures like TTX can then be derived, understood and extended starting from the general risk estimation, making them interpretable in terms of their constituting parameters.**

I. Introduction and Problem Statement

Different risk estimation approaches have been introduced in the last years for the support of behavior planning of intelligent Advanced Driving Assistance (ADAS) and Autonomous Driving (AD) systems. Most prominent are the Time-To-Event (TTX) based approaches, like Time-To-Collision (TTC) which are defined as the deterministic time remaining until the trajectories of two traffic participants intersect, see e.g. [1], and which seem to correlate to some extent with human perceived risk from optical flow expansion patterns. Time-Headway (TH) is another popular risk measure, in the sense that e.g. TTC and TH statistics have been used to analyze the relative risk at special road areas, assuming that small TTC's or headways are relatively unsafe. Several extensions and variations of the TTX idea exist, including Post-Encroachment Time (PET), Time-To-Brake (TTB) [2], Time-To-React (TTR) (see e.g. [3], [4]), "Worst-TTC" [5], Time-Exposed-Time-To-Collision (TET), and Time-Integrated-Time-To-Collision (TIT) [6]. For collision risk estimation, methods based on Gaussian state distributions have been used, see e.g. [7], [8]. Common to most of these risk measures is that they correlate with criticality, but that they are only loosely connected to the underlying microscopic phenomena in terms of prediction, uncertainties, state statistics, and scene evolution. In this paper, we present a unified framework for risk estimation based on microscopic models and which generalizes several of the previous risk measures.

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A. Prediction, Instantaneous Risk and Integral Risk

Risk is essentially the potential resp. the expectation of a loss or a negative occurrence in the future. It involves an element of uncontrollability or uncertainty, because it is by definition probabilistic: We are not completely certain if a risk will occur, i.e., there is usually the possibility that it might not happen. For safe driving using intelligent ADAS and AD technologies, risk estimation is therefore one of the most crucial elements (see e.g. [9]), since safe driving involves risk prediction and the support and evaluation of strategies that lead to the avoidance of the risky event or the mitigation of its consequences.

Mathematically, risk is the combination of the probability that something detrimental will happen (event occurrence probability), combined with a cost in case that an event happens (severity), so by construction, risk estimation involves two components. In publications related to risk estimation, the terminology is often such that risk is treated as equivalent to occurrence probability (i.e., the severity term is neglected or set to a constant cost per event), so that e.g. when we talk about "collision risk" or "accident risk", we actually address collision probability or accident probability. This is because the occurrence probability is usually more challenging to model, and the starting point for risk-avoiding behavior planning strategies. This paper also concentrates on the analysis of models for the estimation of occurrence probability, however, severity can be easily incorporated.

Because risk involves hypothetical events that have not yet happended, inherent to all risk estimation approaches is an assumption about how a scene will evolve in the immediate future. This assumption is sometimes not really explicit, as it is the case for most TTX-based risk measures (see section I-B). In this paper, for clarity of comparisons, we will always state the prediction models explicitly. However situation prediction is only one necessary component, following a schema based on the steps of (i) situation assessment including current state estimation, (ii) situation prediction based on state and uncertainty propagation, (iii) instantaneous risk estimation based on the predicted states, and (iv) integral risk estimation.

We start with a traffic scene at current time t involving traffic elements / traffic participants j with their states $\mathbf{x}_j(t)$.

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From the current time, we make a prediction of the scene for future times $s \in]0, s_{\max}]$, meaning that we estimate expected future states $\hat{\mathbf{x}}_j(t+s)$, according to some **situation-specific prediction** model (e.g., vehicles continue driving with constant velocity along a path). We are interested e.g. in the risk for an entity i, so from these states, for each predicted future time step $t+s$ we estimate risk indicators $I_i(\{\hat{\mathbf{x}}_j(t+s)\}_j)$. As an example, we might look at the predicted positions $\hat{\mathbf{r}}_i(t+s)$ and $\hat{\mathbf{r}}_j(t+s)$ to check if the two traffic participants overlap with each other at the future time $t+s$, which is an indicator for a collision event.

Based on the risk indicators, we calculate an **instantaneous risk** for each predicted time step $t+s$ (more precisely, we calculate an instantaneous risk for a small time interval of length Δt around $t+s$). Once we have the instantaneous risk function over the entire prediction time, we can use it to calculate an **integral risk** $R(t)$ that our entity is exposed to at time t, and which contains the full expectation of critical events that will happen in the future. In our derivations in section I-D we will explain the mathematical formalism that details each of these steps.

Two aspects deserve attention here. First of all, a prediction of future states is usually not deterministic, since there are uncertainties involved. As an example, the state estimations $\mathbf{x}_j(t)$ at prediction start might be derived from sensors and thus subject to measurement errors (measurement uncertainty). Alternatively, the driver of the vehicle might exhibit speed variations, accelerating or decelerating (behavioral uncertainty). To account for the uncertainties, instead of looking at the states $\hat{\mathbf{x}}_j(t+s)$ we model the set of possible states by probability distribution functions (**pdf**'s) which quantify the probability (density) that at the future time $t+s$, certain states will be adopted. As a simple example, we might model the states as being Gaussian distributed, so that $\mathbf{x}_j \sim \mathcal{N}(\mathbf{x}_j|\hat{\mathbf{x}}_j, \Sigma_{\mathbf{x},j})$, with mean $\hat{\mathbf{x}}_j$ and covariance matrix $\Sigma_{\mathbf{x},j}$. This applies both for current (t) states pdfs (incl. measurement uncertainties) as well as for predicted states pdf's (incl. behavior uncertainties), for which appropriate uncertainty propagation models, like e.g. for $\hat{\mathbf{x}}_j(t+s)$ and $\Sigma_{\mathbf{x},j}(t+s)$, have to be provided.

B. Classical Risk Measures of TTX Type

For risk estimation based on Time-To-Collision (TTC) indicators (e.g. [1]), the prediction model is such that it is assumed that a vehicle drives along a longitudinal path, followed by a faster other vehicle along the same path, and we measure the time until a follower car collides with a car in front. The vehicles drive with constant velocities so that given the prediction model of longitudinal traveled distance starting at t

$$r_j(t+s) := r_j(t) + v_j\, s \tag{1}$$

and position and velocity differences

$$\begin{aligned} \Delta r_{ij}(t+s) &:= r_j(t+s) - r_i(t+s) \\ \Delta v_{ij} &:= v_j - v_i \end{aligned} \tag{2}$$

we get $\Delta r_{ij}(t+s) = \Delta r_{ij}(t) + \Delta v_{ij} s$ and the Time To Collision (TTC)

$$s^*_{ij}(t) := -\frac{\Delta r_{ij}(t)}{\Delta v_{ij}} \quad , \tag{3}$$

i.e., $t + s^*_{ij}(t)$ is the time at which a crash with $\Delta r_{ij} = 0$ is predicted to occur. From the signs, this can be certain to happen as long as $s^*_{ij}(t) \geq 0$. For generality, we have omitted vehicle dimensions here, however they can be easily included by e.g. subtracting a car length L_{ij} from Δr_{ij}. (In the following, for the sake of brevity, we will omit all the indices and dependencies and write only s^* for $s^*_{ij}(t)$.)

Time Headway (TH) is the time difference between consecutive traffic participants to pass a fixed point in space. In terms of prediction models, it is "as if" the leading car would take an immediate full stop to velocity 0 whereas the following car continues to drive with its constant velocity, and then we measure the time it lasts until the following car collides with the leading car. We will therefore refer to it as $\text{TTC}_{\text{c,0}}$, since it is a special case of TTC (i.e., it is a case of TTC with a special prediction model for the leading car).

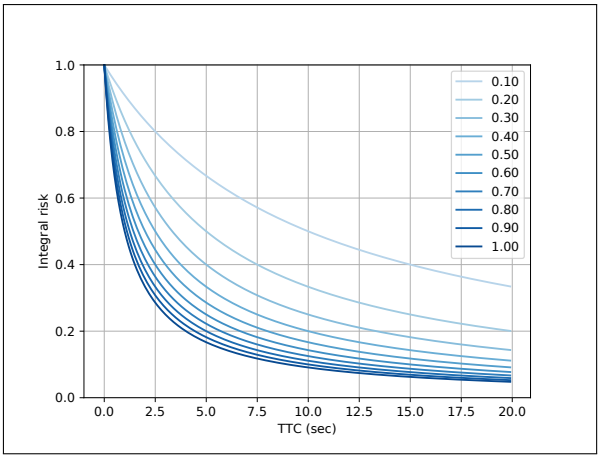

Fig. 1: Heuristic risk function $R_{\text{TTC},ij}$ for TTC scenarios, interpretable as 2-entity collision probability $\in [0,1]$ as function of TTC s^*, plotted for different parameter values of $\bar{\tau}^{-1}_{\text{TTC}}$. It is assumed that for larger remaining time s^* until the collision, there is an increasing probability to avoid the accident. As s^* decreases, the collision becomes imminent and finally, unavoidable so that $R_{\text{TTC},ij} \to 1$. At the other end, when $s^* \to \infty$, the risk vanishes and $R_{\text{TTC},ij} \to 0$.

How do we then derive risk measures from TTX calculations? Considering the TTC case, the heuristic assumption is that the overall risk of a predicted collision event at a time $t+s^*$ in the future decreases if there is more time s^* left until the collision happens. The reason behind this decrease can be manifold; one assumption is that if there is more time left until the event occurs, there is also a larger chance for ego-driving behaviors to happen which might lead to a deviation from an assumed trajectory and ultimately to an avoidance of the collision (this would be an implicitly assumed behavioral uncertainty).

For the estimation of a collision risk from TTC, we would then set $R(t) \sim 1/s^*$, with the risk $R(t)$ decreasing with larger s^*. However, we would like to interpret this as a collision probability, so we include a normalization factor $\bar{\tau}_{\text{TTC}}^{-1}$ and arrive at

$$R_{\text{TTC},ij}(t) := \frac{1}{1 + \bar{\tau}_{\text{TTC}}^{-1} s^*} \quad , \tag{4}$$

with the time-dependency hidden in $s^* = s_{ij}^*(t)$. Considered as a function of s^*, it starts at 1 for $s^* = 0$ (meaning a collision is certain) and decreases towards 0 for $s^* \to \infty$ with the constant $\bar{\tau}_{\text{TTC}}^{-1}$ controlling the decrease [1]. Fig. 1 shows $R_{\text{TTC},ij}$ for different $\bar{\tau}_{\text{TTC}}^{-1}$, and it can be seen that $\bar{\tau}_{\text{TTC}}^{-1}$ controls the slope of the risk decrease with s^*.

More complex TTX risk measures like e.g. Time-To-Last-Braking rely on variations of this schema. One possibility is to vary the prediction models, such as predicting the car in front to brake down to full stop with constant deceleration with the following car also braking, but with an additional reaction time, during which even an acceleration can still occur. Recently proposed models like the Responsibility-Sensitive Safety Model (RSS, [10]) are examples of these variations.

C. Towards a Unified Risk Theory

There are, however, a series of shortcomings and drawbacks of this type of risk measures. As we have seen, their assumptions about the predicted behavior are rather implicit (TTC, TH) or conservative (TH, TTB, RSS) and do not allow extensions towards arbitrary prediction models in a straightforward manner. In addition, they only work in specific settings (one other traffic participant, 1D longitudinal following, faster follower so that there actually would be a collision), so that for the ADAS and AD handling of other circumstances (serveral traffic participants, lateral risk, intersections, parallel passing and overtaking, rule violation), further specific risk models have to be integrated. This then poses the question of risk combination: In scenarios with several different risk sources and risk types, how do we combine or prioritize them, e.g. by maximum probability, minimum time to occurrence, or otherwise? And last, they deal with uncertainties in a completely opaque way (as in the TTC case), so that an understanding and an interpretation of risk models in terms of basic parameters becomes difficult.

Given heuristic risk functions like $R_{\text{TTC},ij}$, the question then becomes: How can we derive and estimate a full-fledged risk model from first principles? What we search for is the true, naturally normalized function that quantifies the entire future expected event occurrence probability (e.g. collision) for a given, general scene evolution. In addition, we target at a risk estimation framework that can be seamlessly extended towards cases not covered by TTX, such as the superposition of several single risks into one integral risk, non-colliding trajectories, other prediction models and 2D scenarios with lateral, passing or intersection behaviors.

[1]Constants designated by overbars.

In the following, to address these questions we will introduce a general survival theory which is the basis for the grounded probabilistic calculation of integral risks with uncertainties. We will then analyze the integral risks and show how the TTX-type risk decay can be derived from explicit behavior uncertainty assumptions, so that TTX risk measures can be formulated and understood in terms of basic risk parameters. Finally, we show that the general survival theory is applicable in a similar form to other scenarios, enabling a natural extension of the tradional TTX risk estimators towards arbitrary prediction models and 2D scenarios.

D. Survival-Theory-Based Risk Measures

Here we shortly summarize parts of the general survival theory (GST, see e.g. [11], [12]) necessary for the derivation of integral risk measures, since an analysis has shown that survival-based risk measures have the potential to provide earlier detection times of critical events and less false positive detections in near-crash and non-crash scenarios [13].

Let us denote the predicted probability that the vehicle i *survives* at a point in time $t' > t$, given that we start our prediction at t, with the **survival probability** $P_{\text{surv},i}(t'|t)$, with starting condition $P_{\text{surv},i}(t|t) = 1$. What is then the probability $P_{\text{surv},i}(t''|t)$ that vehicle i will still survive at a later moment in time t''? This is in essence what survival theory is about. From probability considerations we easily see that

$$P_{\text{surv},i}(t''|t) = P_{\text{surv},i}(t''|t')\, P_{\text{surv},i}(t'|t) \tag{5}$$

and so on. If we split the interval $[t, t'']$ into ever smaller subintervals with corresponding survival probabilities $P_{\text{surv},i}$, we see that $P_{\text{surv},i}(t''|t)$ essentially depends on all time steps within $[t, t'']$, i.e., it depends on the entire prediction time course from t to t''. Written as $P_{\text{surv},i}(t + s|t)$, we get a survival function that starts at 1 for $s = 0$ and which is monotonously always decreasing for increasing s, i.e., it can never grow. In addition, for larger risks (e.g., collisions), it will decrease faster. So how can we calculate $P_{\text{surv},i}$?

Starting from discrete time intervals of size Δt, and looking exclusively at collision events for the survival calculation (other risks can be incorporated without loss of generality into these considerations, see e.g. [14] for a general risk theory based on survival probability), we now introduce $P_{\text{coll},ij}(t'; \Delta t)$ as the collision probability during the time interval $[t', t' + \Delta t]$. We call this the **instantaneous event probability** for collisions, since it will depend only on the states around t' and not on the full time course $[t, t']$ since prediction start t, in the sense that it will be zero if the states of the entities at t' are such that no collision takes place. For the ego car i, assuming statistical independence of the events, the total probability of not colliding with any other entity during the next short time interval $[t, t + \Delta t]$ is then given by

$$P_{\text{noncoll},i}(t; \Delta t) = \prod_{j \neq i} [1 - P_{\text{coll},ij}(t; \Delta t)] \tag{6}$$

and consequently the probability of colliding by

$$P_{\text{coll},i}(t;\Delta t) = 1 - P_{\text{noncoll},i}(t;\Delta t) \tag{7}$$

The probability that vehicle i will not be engaged in any of the events during the next few discrete time intervals, i.e., the probability that the vehicle survives one, two or n steps further, can be expressed by

$$\begin{aligned} P_{\text{surv},i}(t+\Delta t|t) &= P_{\text{noncoll},i}(t;\Delta t)\, P_{\text{surv},i}(t|t) \\ P_{\text{surv},i}(t+2\Delta t|t) &= P_{\text{noncoll},i}(t+\Delta t;\Delta t)\, P_{\text{surv},i}(t+\Delta t|t) \\ P_{\text{surv},i}(t+n\Delta t|t) &= \dots \end{aligned} \tag{8}$$

(starting with $P_{\text{surv},i}(t|t) = 1$) so that we recursively get

$$P_{\text{surv},i}(t+n\Delta t|t) = \prod_{k=0}^{n-1} P_{\text{noncoll},i}(t+k\Delta t;\Delta t)\, P_{\text{surv},i}(t|t) \quad . \tag{9}$$

For small time intervals $\Delta t \to 0$ we can set

$$P_{\text{coll},ij}(t';\Delta t) \approx \frac{d}{dt} P_{\text{coll},ij}(t';\Delta t)\,\Delta t := \tau^{-1}_{\text{coll},ij}(t')\,\Delta t \tag{10}$$

with the predicted collision **event rate** $\tau^{-1}_{\text{coll},ij}$ (number of collision events between entity i and j per time unit), which then leads to the total event rate (specific for entity i)

$$\tau_i^{-1}(t') = \sum_{j\neq i} \tau^{-1}_{\text{coll},ij}(t') + \text{ other risk rates} \tag{11}$$

and, in continuous time, from eqs. (9), (11) to the survival function

$$P_{\text{surv},i}(t'|t) = \exp\left\{ -\int_t^{t'} \tau_i^{-1}(t'')\, dt'' \right\} \quad . \tag{12}$$

These last two equations can be generalized to arbitrary other risks quantifiable via risk event rates, with all risk event contributions adding linearly as indicated in eq. (11). The event rates are functions that depend on all the predicted states $\hat{\mathbf{x}}_j(t')_j$ through risk indicators $I_i(\{\hat{\mathbf{x}}_j\}_j)$ like e.g. the future expected overlap of the positions of traffic participants.

With the survival function (either calculated discretely eq. (9) or continuously eq. (12)), it is straighforward to calculate the real, **time-course sensitive (tcs)** predicted risk for each event contribution, in our case e.g. the collision with j, according to

$$P_{\text{tcscoll},ij}(t'|t,\Delta t) = P_{\text{coll},ij}(t';\Delta t) P_{\text{surv},i}(t'|t) \quad , \tag{13}$$

which is the probability that, starting at time t, the car i gets involved in a collision event at some predicted time t' and does not get involved in any other event before during $[t,t']$. For this to happen, the vehicle i first has to survive potential collisions with all entities until t', to then get involved in a collision with entity j during the interval of size Δt around t'.

A beneficial side effect is that we can use eq. (13) to calculate the integral event probabilities and risk contributions (see e.g. [12]) that the ego-vehicle is exposed to under given predicted trajectories, simply by integrating over the future time,

$$R_{ij}(t) := \sum_n P_{\text{tcscoll},ij}(t+n\Delta t|t,\Delta t) \quad , \tag{14}$$

or, expressed continuously,

$$\begin{aligned} R_{ij}(t) &:= \int_t^\infty \frac{P_{\text{tcscoll},ij}(t'|t,\Delta t)}{\Delta t}\, dt' \\ &= \int_t^\infty \tau^{-1}_{\text{coll},ij}(t')\, P_{\text{surv},i}(t'|t)\, dt' \quad . \end{aligned} \tag{15}$$

This is the total probability (for $\Delta t \to 0$ it can be shown that $R_{ij}(t) \in [0,1]$) that vehicle i will get involved in a collision with entity j at some point in the future. It obeys correct normalization in the sense that added over all other risk events the total probability is 1, meaning that $R_{ij}(t)$ gives us the contribution of the risk involved with a specific collision i,j. The normalization to 1 means that on the long run, one of the events will certainly kick in (this includes special "escape" events which will get important in the next section). By addition of the single integral event probabilities, we can get more unspecific integral probabilities, e.g. by adding $R_{ij}(t)$ over all other cars $j \neq i$, we get the probability that car i *will be engaged in a collision event with any other car*.

$$R_i(t) = \sum_{j\neq i} R_{ij}(t) + \text{other integral risks} \tag{16}$$

Summarizing, the survival theory allows to estimate the instantaneous event risks as well as the single and total event integral risks over the future time, given a predicted time course of the scene.

II. Derivation of Integral Risk Measures from General Survival Theory (GST)

Now let us consider the TTC scenario from section I-B, where the 2 involved vehicles are predicted to continue driving with constant velocity along a longitudinal path. From section I-D, we have that the survival function for the collision risk between vehicle i and vehicle j is (written here in slighty different form to explicitly see the prediction time)

$$P_{\text{surv},i}(t+s|t) = \exp\left\{ -\int_0^s \tau_i^{-1}(t+s')\, ds' \right\} \tag{17}$$

with a time-differential event rate expressed by the 2-car collision events so that $\tau_i^{-1} = \tau^{-1}_{\text{coll},ij}$, according to eq. (11).

What would then be a good collision event rate as a function of time for this scenario? It is reasonable to assume that $\tau^{-1}_{\text{coll},ij}$ starts with 0 at very far times, increases when the collision event comes proximal, and approaches ∞ when $\Delta r_{ij}(t+s^*) = 0$. A function of this type would be

$$\tau^{-1}_{\text{coll},ij}(t+s) := \frac{\alpha^{-1}}{(s^* - s)} \quad , \tag{18}$$

where e.g. for $\Delta r_{ij}(t) > 0$ (vehicle j in front of vehicle i) it must be $\Delta v_{ij} < 0$ (slower vehicle in front approached by rear car) so that the collision time lies $s^* \geq 0$ in the future, and α^{-1} is a constant parameter without units.

Inserting this into the event rate $\tau_{\text{coll},ij}^{-1}$ of the survival function eq. (17), we can integrate which leads to

$$\begin{aligned} P_{\text{surv},i}(t+s|t) &= \exp\left\{-\int_0^s \tau_{\text{coll},ij}^{-1}(t+s')\,ds'\right\} \\ &= \exp\left\{-\int_0^s \frac{\alpha^{-1}}{(s^*)(1-s'/s^*)}\,ds'\right\} \\ &= \exp\left\{-(\alpha^{-1})\left[-\log(1-x)\right]\Big|_0^{s/s^*}\right\} \\ &= (1-s/s^*)^{(\alpha^{-1})} \quad , \end{aligned} \tag{19}$$

valid for $s \leq s^*$.

At the collision time $t+s^*$, this results in $P_{\text{surv},i}(t+s^*|t) = 0$ as expected (and for $s > s^*$, it is $P_{\text{surv},i}(t+s|t) = 0$ by construction). What is the event probability then? Using eqs. (10), (13) and (19) we get that

$$\begin{aligned} P_{\text{tcscoll},ij}(t+s|t,\Delta t) &= \Delta t\, \tau_{\text{coll},ij}^{-1}(t+s) P_{\text{surv},i}(t+s|t) \\ &= \alpha^{-1}(\Delta t/s^*)\,(1-s/s^*)^{(\alpha^{-1}-1)} \quad . \end{aligned} \tag{20}$$

The integral risk is then

$$\begin{aligned} R_{ij}(t) &= \int_0^\infty \frac{P_{\text{tcscoll},ij}(t+s|t,\Delta t)}{\Delta t}\,ds \\ &= \int_0^{s^*} \alpha^{-1}(1/s^*)\,(1-s/s^*)^{(\alpha^{-1}-1)}\,ds = 1 \ , \end{aligned} \tag{21}$$

meaning that strictly speaking, the risk of ending up in a collision state is 1, which is indeed the case for this scenario.

III. General Integral Risk

A. General Integral Risk for the TTC Scenario

The last finding of section II is at odds with the interpretation of 1/TTC as an indicator for the decreasing integral collision risk $R_{\text{TTC},ij}$ in this scenario. However, with a slight extension, we get a much more interesting derivation. The problem is that for eqs. (13) and (19) we have not taken the ego-behavior uncertainty into account. In a real world setting, there is a chance that the ego-vehicle i will not follow strictly the predicted trajectory. This might be longitudinally due to subtle braking or acceleration, either by actuation or by motor variability, but it could also be caused by steering and avoidance maneuvers with lateral components whose impact amplifies with prediction time s. In some of these cases, the ego-vehicle will avoid the target, i.e., it will *escape* from its prediction trajectory. We assume this effect to endure during the entire driving time, and quantify it by an **escape rate** $\bar{\tau}_{\text{esc}}^{-1}$, which captures the probability density that such an event will happen. Following eq. (11) this then leads to $\tau_i^{-1} = \bar{\tau}_{\text{esc}}^{-1} + \tau_{\text{coll},ij}^{-1}$ and consequently to

$$\begin{aligned} P_{\text{surv},i}(t+s|t) &= \exp\left\{-\int_0^s [\bar{\tau}_{\text{esc}}^{-1} + \tau_{\text{coll},ij}^{-1}(t+s')]\,ds'\right\} \\ &= (1-s/s^*)^{(\alpha^{-1})}\ e^{-\bar{\tau}_{\text{esc}}^{-1}\,s} \quad . \end{aligned} \tag{22}$$

At each point in time during prediction, the vehicle then has 2 options: Either it is hit by a collision event, or it undergoes an escape event. The latter case is a sort of collector class for all types of uncertain events other than explicit risks like collision. In case of escape events, we assume the further trajectory to deviate disruptively from the predicted trajectory. The approach therefore models all possible risky events explicitly using corresponding event rate functions but regards all escaping vehicles as safe, or at least as not being involved in any of the risky events.

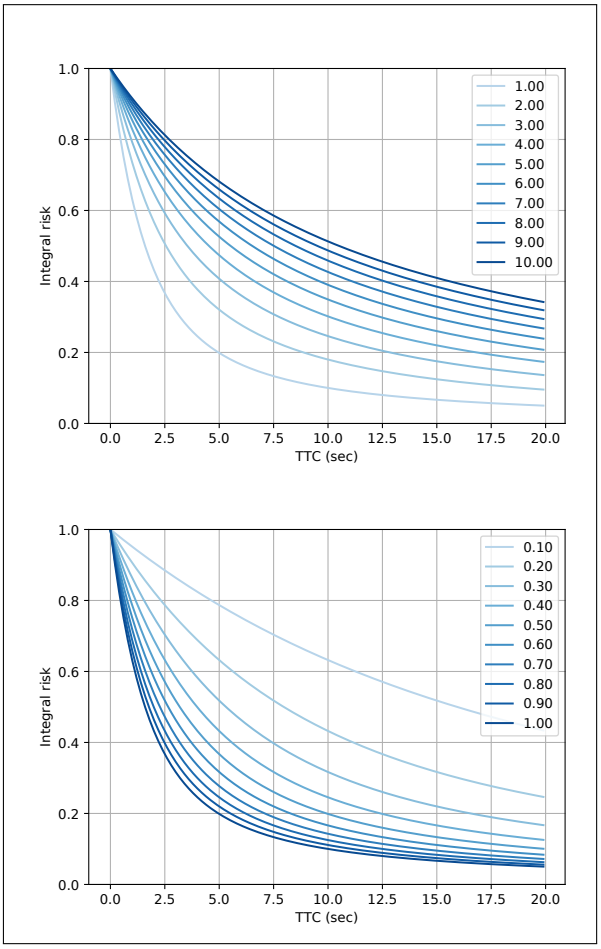

Fig. 2: Integral collision risk R_{ij} plotted as function of s^* for TTC scenarios calculated from the general survival theory. Top: Fixed $\bar{\tau}_{\text{esc}}^{-1} = 1.0$ Variation of the parameter α^{-1}. Bottom: fixed $\alpha^{-1} = 1.0$, variation of the parameter $\bar{\tau}_{\text{esc}}^{-1}$. A larger escape rate $\bar{\tau}_{\text{esc}}^{-1}$ decreases risk sensitivity, whereas a larger collision rate factor α^{-1} increases it.

The event probability for collision events only is then given by considering the vehicle to survive all events (including escape events) until $t+s$, meaning that the vehicle continues on the predicted trajectory, and then being subject to a collision event, so that we have

$$\begin{aligned} P_{\text{tcscoll},ij}(t+s|t,\Delta t) &= \Delta t\, \tau_{\text{coll},ij}^{-1}(t+s) P_{\text{surv},i}(t+s|t) \\ &= \alpha^{-1}(\Delta t/s^*)\,(1-s/s^*)^{(\alpha^{-1}-1)}\ e^{-\bar{\tau}_{\text{esc}}^{-1}\,s} \quad . \end{aligned} \tag{23}$$

We again calculate the integral risk for time t by integrating over the entire future $t+s$, which results in the overall risk of getting involved in a collision event with vehicle j for this scenario. This probability is now smaller than 1, because

escape events are likely on the trajectory towards the impact. In addition, the exponential escape term reduces the overall collision risk further for larger times s^* that remain until the impact, because there is a longer period of time where escape events can occur and thus the collision can be avoided.

As a result, we get a collision risk which naturally decreases for increasing TTC s^*, fully derived from probabilistic considerations taking escape behaviors and collision probability uncertainty into account. The integral risk amounts to

$$\begin{aligned} R_{ij}(t) &= \int_0^{\infty} \frac{P_{\text{tcscoll},ij}(t+s|t,\Delta t)}{\Delta t}\, ds \\ &= \int_0^{s^*} \alpha^{-1}(1/s^*)\,(1-s/s^*)^{(\alpha^{-1}-1)}\; e^{-\bar{\tau}_{\text{esc}}^{-1}\, s}\, ds \\ &= \int_0^{1} \alpha^{-1}\,(1-x)^{(\alpha^{-1}-1)}\, e^{-\beta^{-1} x}\, dx \end{aligned} \tag{24}$$

with unitless parameters α^{-1} and $\beta^{-1} := \bar{\tau}_{\text{esc}}^{-1} s^*$ and the time-dependency via $s^* = s_{ij}^*(t)$. This integral risk is a full-fledged probability (the probability that the ego-vehicle will encounter a collision at any time ahead, predicted from states starting at time t) and correspondingly it is always normalized: $0 \leq R_{ij}(t) \leq 1$.

From the previous section we know that for the TTC scenario an integral risk exists, is bounded, and represents a true collision probability derived from a first principles risk theory involving as interpretable parameter the instantaneous collision event rate constant $\bar{\tau}_{\text{esc}}^{-1}$.

Expressed by eq. (24), the integral risk $R(t)$ is a function of s^* (the time until a collision will happen in predicted time s) through the parameters α and β. Through s^*, it is also indirectly a function of t, since s^* depends on $\Delta r_{ij}(t)$ and $\Delta v_{ij}(t)$.

As a function of s^*, eq. (24) starts at 1 for $s^* \to 0$, as we would expect if the impact is unavoidable. It decreases for larger s^*, and descends to 0 for $s^* \to \infty$. In the next section, we will see why this is a good assumption and how it can be derived from further microscopic considerations. In the simplest of cases, when $\alpha = 1$, the integral simplifies to

$$R_{ij}(t) = \frac{1}{\bar{\tau}_{\text{esc}}^{-1}\, s^*}\left(1 - e^{-\bar{\tau}_{\text{esc}}^{-1}\, s^*}\right) \quad , \tag{25}$$

again with the time-dependency through $s^* = s_{ij}^*(t)$. We can compare this risk with the heuristic risk function eq. (4) that we set up previously for TTC in Fig. 2. Note the qualitative match, especially for $\bar{\tau}_{\text{esc}}^{-1} = \bar{\tau}_{\text{TTC}}^{-1}$. Fig. 3 finally shows the collision risk function $R_{ij}(t)$, but now as a function of time t, while the TTC situation evolves. At each time t, this amounts to predicting the future scene evolution, calculating the survival function over the predicted time according to eq. (12), and integrating the instantaneous time-course-sensitive risk to an integral risk (15) and (16).

B. Extension to 2D Integral Risk

To the contrary of the TTC risk probability (eq. (4)), we have derived the integral risk from microscopic considerations like the escape rate and the instantaneous collision rate,

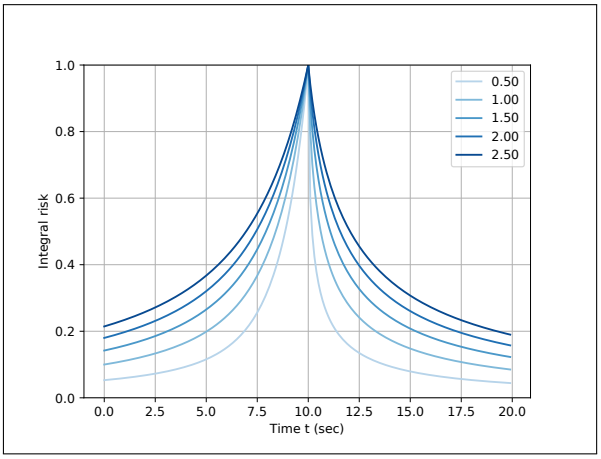

Fig. 3: Integral collision risk $R_{ij}(t)$ as a function of time t, for different parameters α^{-1}, while the TTC situation evolves. The velocity difference was such that the collision time is at 10 sec after start.

as well as the measured and predicted states of the involved traffic participants. This approach can be extended towards other risk types, like collision with road infrastructure, speeding and rule violation risks, as well as curve risks, as long as appropriate state-dependent instantaneous risk rates can be formulated.

Reconsidering in detail the approach for the instantaneous collision probability from III-A, we might also be able to derive integral risk probabilities that are valid for general scenarios (including transversal, overtaking and near-miss cases), and not only if a collision is certain along a predicted longitudinal path. This would mean that we would have derived a risk probability calculation that is qualitatively encompassing TTX based risk estimators for special cases but that can be applied to arbitrary other scenarios as well.

Let us start from a probabilistic description of the states of two vehicles involved in a hypothetical collision event. In this case, the position of each might be described by a Gaussian probability distribution. It is interesting then to estimate the probability distribution $p(\Delta \mathbf{r}_{ij})$ of their relative 2D position difference $\Delta \mathbf{r}_{ij}$ to each other. A collision would then occur, if the two vehicles' geometries overlap, i.e., if $\Delta \mathbf{r}_{ij}$ lies within a certain area Ω determined by the physical geometries of the vehicles, with Ω extending around $\Delta \mathbf{r}_{ij} = 0$ where both vehicles are located on top of each other. The integration of $p(\Delta \mathbf{r}_{ij})$ over the area Ω where the vehicles overlap results in a total probability, given the uncertainties. This is a standard approach in collision risk estimation, see e.g. [7].

However, the overlap probability itself is not as relevant as the *change of overlap probability* due to the physical motion of the vehicles. Roughly, if the extension of the distribution $p(\Delta \mathbf{r}_{ij})$ is larger than the area Ω, we can set $p(\Delta \mathbf{r}_{ij}) \approx p(0) \approx$ const inside Ω. We then would get that the absolute change of overlap probability is approximately equal to the flux of probability density over the borders of area Ω [7], so

that

$$P_{\text{coll},ij} \approx \alpha^{-1} \, |\Delta v_{ij}| \, \Delta t \; p(\Delta \mathbf{r}_{ij}) \quad . \tag{26}$$

On the other hand, if the uncertainties are small, the extension of the distribution $p(\Delta \mathbf{r}_{ij})$ also gets very small and diverges, which we model by

$$p(\Delta \mathbf{r}_{ij}) \sim \frac{1}{|\Delta \mathbf{r}_{ij}|} \tag{27}$$

Putting together eqs. (26), (27) and (10), for the 1D position we get for the predicted collision event rate

$$\tau^{-1}_{\text{coll},ij}(t+s) = \frac{\alpha^{-1} \, |\Delta v_{ij}(t+s)|}{|\Delta r_{ij}(t+s)|} \tag{28}$$

and in the TTC case

$$\begin{aligned} \tau^{-1}_{\text{coll},ij}(t+s) &= \frac{\alpha^{-1} \, |\Delta v_{ij}|}{|\Delta r_{ij}(t) + \Delta v_{ij} s|} \\ &= \frac{\alpha^{-1}}{|s^* - s|} \end{aligned} \tag{29}$$

which for $0 \leq s \leq s^*$ is of the same form as eq. (18). Here, we have written $|s^* - s|$ to indicate that the function remains valid in regimes $s > s^*$ if the integral risk is less than 1, as we will see later in an example of lateral passing.

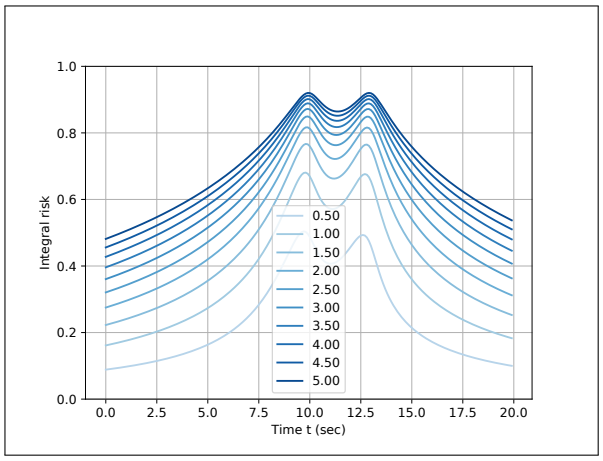

Fig. 4: Integral collision risk $R_{ij}(t)$ in a parallel passing scenario for varied parameter α^{-1}. The ego car passes two cars in front, but on a parallel lane. The integral risk increases, but stays below 1 because of the parallel offset. Parameters: $\bar{\tau}^{-1}_{\text{esc}} = 1/\text{sec}$, $\Delta r_{ij}(0) = \{100, 130\}$ m, $\Delta v_{ij}(0) = \{-10, -10\}$ m/sec, lateral passing offset 5 m.

From eq. (29) we can see that the approach (18) was indeed justified. But as an important side note, we also see from the first line of eq. (28) that we now get a formalism that works for other scenarios: It does not depend on the assumption of constant velocity (any trajectory predictions can be used), and in general 2D form, we can use the 2D extension

$$\tau^{-1}_{\text{coll},ij}(t+s) = \frac{\alpha^{-1} \, ||\Delta \mathbf{v}_{ij}(t+s)||}{||\Delta \mathbf{r}_{ij}(t+s)||} \tag{30}$$

in conjunction with the survival formalism from section I-D and calculate the integral risk according to eq. (15).

As an example, let us consider a scenario similar to the TTC approach: The close passing-by of another vehicle in a neighboring lane, with parallel velocities. A TTC-based risk measure would have to take a yes/no decision if there could be a longitudinal collision. If yes, the normal TTC formalism would apply with full risk consideration, but otherwise the risk would be zero. Instead, it is clear that close passing might impose a considerable risk, which depends gradually on the passing distance. Separating $\Delta \mathbf{r}_{ij}$ into longitudinal and lateral components Δr_{ij} and d_{ij} (let us assume that we drive with constant velocity differences on a longitudinal path with a constant lateral offset d_{ij} to vehicle j) we then would get

$$\begin{aligned} \tau^{-1}_{\text{coll},ij}(t+s) &= \frac{\alpha^{-1} \, |\Delta v_{ij}|}{\sqrt{(\Delta r_{ij}(t+s))^2 + (d_{ij})^2}} \\ &\approx \frac{\alpha^{-1} \, |\Delta v_{ij}|}{d_{ij} + |\Delta r_{ij}(t+s)|^2 / (2\, d_{ij})} \\ &\approx \frac{\alpha^{-1}}{d_{ij}/|\Delta v_{ij}| + |(s^* - s)| \, \Delta r_{ij}(t+s)/(2\, d_{ij})} \end{aligned} \tag{31}$$

where in the last two lines we have assumed that we are close to the maximum approach of the two vehicles and $\Delta r_{ij} \to 0$. It can be seen that the instantaneous event rate is very similar to eq. (29), but with an additional d_{ij}-dominated offset.

As a consequence, the maximal integral risk does not reach 1 for $s^* \to 0$. With increasing lateral distance, the risk also decreases. The integral risk function based on eq. (30) can be seen in Fig. 4, this time plotted as function of time t as the scenario evolves. At $t = 10$ sec, which is the point when the ego vehicle gets closest to the first vehicle in front, the risk is maximal but < 1, so in this case the overall risk resembles approaches where the *point and time of closest encounter* is used to get risk estimates, see e.g. [13].

C. Extension to Multiple Risk Sources

In standard approaches, the normalization of risk measures as required for the interpretation of a probability that a certain event will occur is usually not fulfilled by construction. Instead, it has to be included artificially, e.g. by capping the risks at a maximum, so that the total risk remains normalized within $[0, 1]$. However, this type of normalization can lead to erroneous risk estimations and consecuently, wrong behaviors. E.g., a system could try to avoid a risk that is very large but distant in time and neglect a temporally more proximal risk.

To the contrary, we have seen that in the survival theory approach, we collect all the instantaneous risk rates for the different risk types into a combined rate eq. (11). Then we can calculate the survival probability as indicated in eq. (12). For the longitudinal constant velocity case with multiple slower vehicles in front (multi-TTC case), we get

$$P_{\text{surv},i}(t+s|t) = \prod_j \, |(1 - s/s^*_{ij})|^{(\alpha^{-1})} \, e^{-\bar{\tau}^{-1}_{\text{esc}} \, s} \quad , \tag{32}$$

so that the survival probability reduces to 0 as soon as the most immanent crash with the lowest s^*_{ij} occurs. Neverthe-

less, other traffic participants are seamlessly considered, and it is always $0 \leq R_i(t) \leq 1$ by construction.

In practice, instead of such analytic formulas for the survival function $P_{\text{surv},i}(t+s|t)$ and the integral risk $R_i(t)$ based on s^*, we would rather use directly the event rates like eq. (30), since they relate to the predicted physical states, and calculate $P_{\text{surv},i}(t+s|t)$ following eq. (12) and $R_i(t)$ using eqs. (15) and (16).

To illustrate this let us consider again the passing scenario with two vehicles that are passed. In addition, we include another vehicle that crosses an intersection in front of the ego vehicle. We use the 2D risk rate eq. (30), and arrive at an integral risk as shown in Fig. 5, with 3 peaks. All cars drive at constant velocity, and the ego car first passes the crossing car by a close miss, and then passes the other 2 vehicles on the adjacent lane. Fig. 5 shows the full integral risk as calculated by the combination of all 3 collision risks, and the single risks which would have resulted from scenarios with only one other vehicle, respectively. The full integral risk integrates the vehicle contributions coupled by a common, holistic survival function, which is the reason why it is not a mere addition of the single scenarios (which would all have their own survival function).

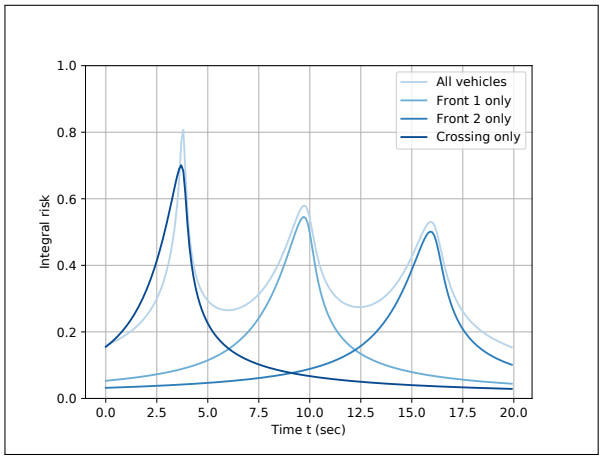

Fig. 5: Integral collision risk $R_{ij}(t)$ for a multiple risk scenario composed of the ego vehicle and 3 other vehicles; one first crossing an intersection and two other ones which are passed on a parallel lane. Parameters: $\bar{\tau}_{\text{esc}}^{-1} = 1/\text{sec}$, $\alpha^{-1} = 0.5$, longitudinal $\Delta r_{ij}(0) = \{100, 130, 80\}$ m, longitudinal $\Delta v_{ij}(0) = \{-10, -8, -20\}$ m/sec, lateral starting position of crossing car 75 m, lateral velocity of crossing car -20 m/sec, lateral passing distances 3 m. The intersection is at 80 m distance, with the crossing car reaching the intersection at 3.75 sec and the ego car at 4.0 sec, then passing the first car in front at 10 sec and the second at 16.25 sec.

IV. Discussion

Starting from general survival theory, we have shown how to derive a probabilistic unified risk estimation model (PURE) for the calculation of integral risks, with the framework being applicable to a number of scenarios. These include arbitrary trajectories and prediction models, as well as 2D and multi-risk extensions. We have shown that "standard" TTX risks can then be considered as special cases.

The most crucial ingredient of the framework are the risk rate functions (e.g. $\tau_{\text{coll},ij}^{-1}$ in eq. (30) for collision). In practical tests with real trajectories (see e.g. [13]), we have found that the exact form of these functions can be varied to some extent, as long as the general dependencies on distances and velocities are maintained. This also enables to include measurement and prediction uncertainties, e.g. by the positions $\mathbf{r}_j(t+s)$ being replaced by the means $\hat{\mathbf{r}}_j(t+s)$ of Gaussian distributions resulting from a state uncertainty propagation model. In this way, probabilistic collision models are fully compatible with the proposed approach. The same holds for different prediction models, like partial acceleration and braking behavior as in the RSS model [10]. Future investigations will include the optimization of parameters that lead to the best event warning statistics and the safest behavior support conditions.

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