Honda Research Institute EU
Honda Research Institute Europe GmbH
https://www.honda-ri.de/

Reference Vector based a posteriori Preference Articulation for Evolutionary Multiobjective Optimization

Ran Cheng, Markus Olhofer, Yaochu Jin

2015

Preprint:

This is an accepted article published in 2015 IEEE Congress on Evolutionary Computation. The final authenticated version is available online at: https://doi.org/[DOI not available]

Reference Vector Based *a posteriori* Preference Articulation for Evolutionary Multiobjective Optimization

Ran Cheng
Department of Computing
University of Surrey
Guildford, Surrey, GU2 7XH
United Kingdom
Email: r.cheng@surrey.ac.uk

Markus Olhofer
Honda Research Institute Europe
Offenbach am Main
Germany
Email: markus.olhofer@honda-ri.de

Yaochu Jin
Department of Computing, University of Surrey
Guildford, GU2 7XH, United Kingdom
College of Information Sciences and Technology
Donghua University, Shanghai 201620, P. R. China
Email: yaochu.jin@surrey.ac.uk

***Abstract*—Multiobjective evolutionary algorithms (MOEAs) usually achieve a set of nondominated solutions as the approximation of the Pareto front. In order to utilize the solutions, a final decision making process is indispensable in most cases in which a small number of solutions have to be selected. In this process a decision maker selects the solutions based on his or her preferences or based on knowledge acquired by observing the approximated Pareto front. Due to the limited number of solutions an algorithm can obtain, in particular when the number of objectives is large, a decision maker may be interested in sampling additional solutions in the preferred region based on the observations of solutions obtained by the algorithm. This paper proposes to use a reference vector based preference articulation (RVPA) method to obtain such additional solutions in preferred regions. After describing the proposed method in detail, experiments are conducted on six benchmark MOPs to assess the performance of the proposed RVPA method. Our empirical results show that, by setting reference vectors in the objective space, the proposed RVPA is able to obtain corresponding solutions in the preferred regions at a much lower cost compared to e.g. a re-start strategy. In addition, by setting the reference vectors in a uniform way, the proposed RVPA method is also able to improve the general quality (convergence and distribution) of the solutions obtained by an MOEA.**

I. Introduction

A multiobjective optimization problem (MOP), which involves several conflicting objectives to be optimized simultaneously, can be formulated as follows:

$$\begin{aligned} \min \quad & \vec{f}(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), ..., f_m(\vec{x})) \\ \text{s.t.} \quad & \vec{x} \in X, \quad \vec{f} \in Y \end{aligned} \tag{1}$$

where $X \subset \mathbb{R}^n$ is the decision space and $\vec{x} = (x_1, x_2, ..., x_n) \in X$ is the decision vector, $Y \subset \mathbb{R}^m$ is the objective space and $\vec{f} \in Y$ is the objective vector, which is composed of m objective functions $f_1(\vec{x})$, $f_2(\vec{x})$,...,$f_m(\vec{x})$ that map $\vec{x}$ from X to Y. Since the objectives often conflict with each other, it is impossible to identify one single decision vector which minimizes all objectives simultaneously. Instead, an arbitrary number of optimal trade-off solutions can be obtained, which form the Pareto front (PF) in the objective space.

In general, since it can be deduced from the Karush-Kuhn-Tucker optimality conditions that the PF is a piecewise continuous manifold [1], [2], enumerating each solution on the PF is impossible. Instead, only a limited number of nondominated solutions can be obtained as an approximation of the true PF. The population-based nature of evolutionary algorithms, multiobjective evolutionary algorithms (MOEAs), as a prevalent technique to obtain such nondominated solutions, have witnessed a boom of development during the past decade [3]. The main focus of MOEAs is to obtain a group of evenly distribute nondominated solutions which are as close to the true PF set as possible. And based on the observations on the distribution of the approximated PF, a decision maker is able to pick up preferred solutions, i.e., do decision making. However, in some practical scenarios, issues may arise when the solutions obtained by an MOEA in a single run may not satisfy the requirement of a decision maker: on the one hand, few or no solutions may be in the preferred regions; on the other hand, the solutions may be of poor quality due to a high distance to the true Pareto front. In order to obtain more dense sampling, a decision maker can either run the similar MOEA again with different parameter settings or try to interpolate the results on the Pareto front, both of which are difficult tasks.

To remedy the issues in the decision making as above, some preference articulation approaches have been proposed with the target to focus the search only on a part of the objective space and thus to increase the density of solutions in this area. These approaches can be roughly categorized into three classes [4]. The first class is known as the *a priori* approaches. A typical example of this class is to use predefined weight vectors to aggregate an MOP into several single objective problems according to a decision maker's preferences [5], [6]. The second class is called the interactive approaches [7], where the preference articulation is repeatedly and progressively specified during the search procedure until a decision maker has found the preferred solutions. The third class is known as the *a posteriori* approaches [8], where the preference articulation is carried out after a set of Pareto op-

timal solutions have already been approximated. It is believed that the *a posteriori* approaches are able to produce more and usually better Pareto optimal solutions in the interested regions specified by a decision maker [9].

This paper proposes to use reference vectors in the objective space for preference articulation *a posteriori*. In the proposed method, the reference vectors are defined by a decision maker according to his or her preference, or based on the observations from a given set of Pareto optimal solutions approximated using an existing MOEA. With the reference vectors being defined, the proposed method is able to obtain additional preferred solutions. Meanwhile, if the reference vectors are set in a uniform way, the proposed method is also able to improve the general quality of the solutions approximated by an MOEA.

The rest of the paper is organized as follows. Section II first gives a brief introduction to the reference vectors in the objective space. Then, the proposed reference vector based preference articulation method is introduced in Section III. Section IV presents experimental results. Finally, conclusion is drawn in Section V.

II. Reference Vectors in the Objective Space

Using reference points in the objective space for preference articulation is not new in the literature [10]–[12], while more recently, it is shown that the reference vectors, as a relatively new methodology, are also easy to be defined and used as references in the objective space [13]–[16]. In the following, a brief introduction to the reference vectors in the objective space is given, including methods to set reference vectors, and how to determine the positional relationship between a candidate solution and a reference vector.

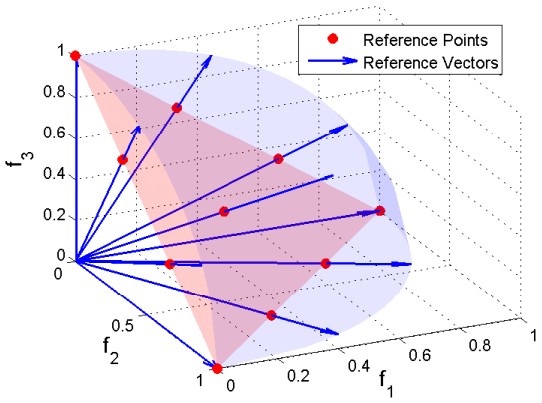


Fig. 1. An illustration of how to generate evenly distributed reference vectors *a priori* in a three-objective space. In this case, 10 uniformly distributed reference points are first generated on a hyperplane, and then they are mapped to a hypersphere to generate the 10 reference vectors.

Reference vectors in the objective space can be set either *a priori* or *a posteriori*. In [15], uniformly distributed reference vectors are predefined *a priori* to cover the entire objective space. In this case, the reference vectors can be transformed from a set of uniformly distributed points by mapping them from a unit hyperplane to a unit hypersphere, as shown in Fig. 1. Specifically, the uniformly distributed points on the unit hyperplane are generated using the canonical simplex-lattice design method [17]:

$$\begin{cases} w_k^i \in \{\frac{0}{H}, \frac{1}{H}, ..., \frac{H}{H}\}, \sum_{i=1}^{m} w_k^i = 1, \\ \vec{w}_k = (w_k^1, w_k^2, ..., w_k^m), k = 1, ..., K, \end{cases} \tag{2}$$

where m is the objective number, K is the number of uniformly distributed points, and H is a positive integer for the simplex-lattice design. Then, the corresponding unit reference vectors $\vec{v}_k$ can be obtained via the following transformation:

$$\vec{v}_k = \frac{\vec{w}_k}{||\vec{w}_k||}, k = 1, ..., K. \tag{3}$$

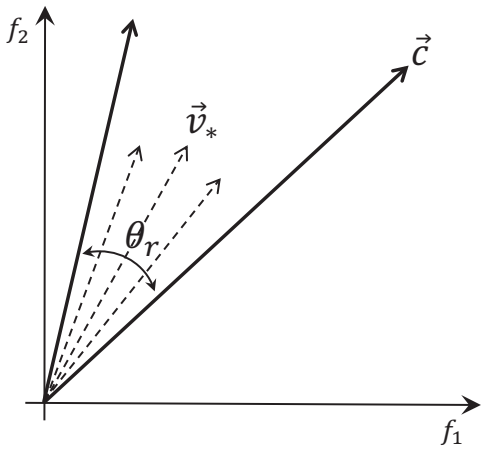


Fig. 2. An example of generating reference vectors *a posteriori* in a preferred region of a bi-objective space. $\vec{c}$ and θ_r represent the center and the scope of the preferred region, respectively, and $\vec{v}_*$ are the reference vectors defined by $\vec{c}$ and θ_r in the preferred region.

Another way to generate the reference vectors is to define them *a posteriori* based on the observations on a given set of Pareto optimal solutions, which is exactly what we have adopted in the proposed preference articulation method. For example, based on the observation of a set of approximate Pareto optimal solutions, a decision maker may only be interested in a small subregion in the objective space. In this case, reference vectors can be generated only inside the preferred subregion. Specifically, such a subregion (denoted as Φ) can be defined by a center vector $\vec{c}$ and a threshold angle θ_r as (refer to Fig. 2):

$$\Phi = \{\vec{v}_* : \theta(\vec{v}_*, \vec{c}) < \theta_r\}, \tag{4}$$

where $\theta(\vec{v}_*, \vec{c})$ denotes the acute angle between any vector $\vec{v}_*$ and the center vector $\vec{c}$. To measure the angles, the corresponding cosine values can be calculated as:

$$\cos\theta(\vec{v}_*, \vec{c}) = \vec{v}_* \times \frac{\vec{c}}{||\vec{c}||}. \tag{5}$$

where $\cos\theta(\vec{v}_*, \vec{c}) > \cos\theta_r$ will indicate $\theta(\vec{v}_*, \vec{c}) < \theta_r$, and vice versa. Similarly, the positional relationship between any

candidate solution (e.g. $\vec{s}_*$) and any reference vector (e.g. $\vec{v}_*$) can be also measure by calculating the cosine value of the acute angle $\theta(\vec{s}_*, \vec{v}_*)$ between them as:

$$\cos\theta(\vec{s}_*, \vec{v}_*) = \frac{\vec{s}_*}{||\vec{s}_*||} \times \vec{v}_*. \quad (6)$$

It should be noted that since the reference vector $\vec{v}_*$ is already a unit vector, no normalization is required in the calculation of the cosine values, while for the center vector $\vec{c}$ and the solution points $\vec{s}_*$, normalization is required in the calculation of the cosine values.

III. The Proposed Preference Articulation Method

Generally, the focus of a preference articulation method is to find Pareto optimal solutions in the preferred regions. In the proposed preference articulation method, the preferred regions are marked with some reference vectors in the objective space as introduced in the previous section, and the aim of the proposed method is to find the Pareto optimal solution that is closest to each reference vector. For simplicity, we term the proposed method as the reference vector based preference articulation (RVPA) method hereafter.

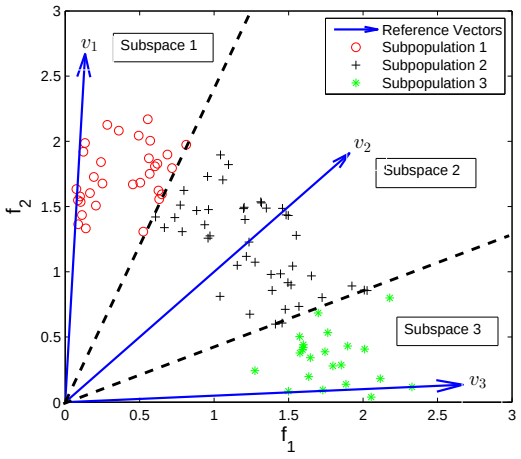


Fig. 3. An example of space division in a bi-objective space. In this case, three reference vectors divide the entire objective space into three subspaces. Correspondingly, the individuals in each subspace are composed of a subpopulation.

As an *a posterior* preference articulation method, RVPA requires a set of approximate Pareto optimal solutions obtained using an existing MOEA as input. As the output, the proposed RVPA generates a new set of approximate Pareto optimal solutions according to the reference vectors which are specified by a decision maker. Specifically, the entire objective space is divided into K subspaces defined by the K reference vectors, and inside each subspace is a subpopulation, as shown in Fig. 3. In each generation, only one individual can be selected from each subpopulation and put into the next generation. In this way, the individuals are able to approach the true PF as well as the reference vectors, and eventually, the final population is expected to contain a set of Pareto optimal solutions, each of which corresponds to a reference vector.

As shown by the framework in Fig. 4, after *initialization*, in each generation t, *reproduction*, *reference vector based population division* and *reference vector based selection* are iteratively operated until the termination condition is satisfied. In the following subsections, the details of the proposed RVPA will be given step by step.

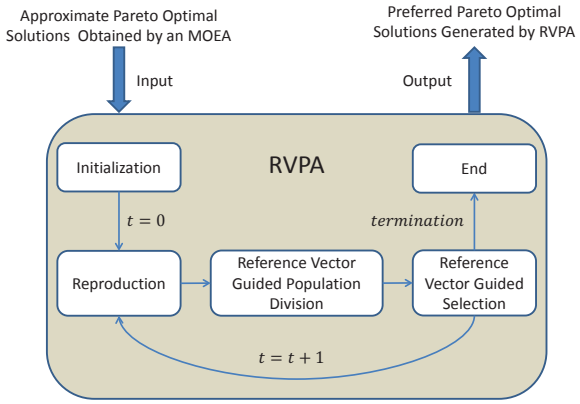


Fig. 4. The framework of the proposed reference vector based preference articulation (RVPA) method.

A. Initialization

To setup the proposed RVPA, the first step is to obtain a set of approximate Pareto optimal solutions (denoted as P^*) using an existing MOEA. Based on the observations on the distribution of P^* in the objective space, a decision maker can set a number of (e.g. K) reference vectors (denoted as $v_i, i = 1, ..., K$) in the objective space according to his or her preferences. Then, we mix up P^* with a number of K randomized individuals to create the initial population $P(0)$. Note that although the size of $P(0)$ is $|P^*| + K$, in later generations, only at most a number of K individuals will be maintained.

B. Reproduction

The SBX crossover and the polynomial mutation are adopted as the reproduction operators in the proposed RVPA, as done by many other MOEAs [3]. The reproduction is operated on the entire population of the current generation $P(t)$. Consequently, the offspring population $Q(t)$, where $|Q(t)| = |P(t)|$, can be generated. Then, we merge $Q(t)$ into $P(t)$ to obtain a merged population, i.e., $P(t) = P(t) \cup Q(t)$.

C. Reference vector based population division

The merged population $P(t)$ is divided into a number of K subpopulations by associating each individual with its closest reference vector. Specifically, the cosine value of the acute angle $\theta_{i,k}$ between each individual $\vec{s}_i$ and each reference vector $\vec{v}_k$ is calculated according to (6), and as a result, $\vec{s}_i$ is associated to $\vec{v}_{k_i}$ iff the θ_{i,k_i} is the smallest among all the angles $\theta_{i,1}, ..., \theta_{i,K}$ between $\vec{s}_i$ and each reference vector.

Note that such a population division strategy does not guarantee a consistent number of individuals in each subpopulation, as the division depends on the distribution of the $P(t)$, which can change dynamically in each generation. However, as we will introduce in the following, the selection strategy is able to guarantee a stable number of individuals in the entire population, regardless of the number of individuals in each subpopulation.

D. Reference vector based selection

As aforementioned, the aim of the proposed RVPA is to obtain the Pareto optimal solutions which are closest to the reference vectors, where each solution corresponds to one reference vector. Therefore, the selection procedure is designed to choose at most one individual from each subpopulation.

To begin with, the non-dominated sorting is carried out to identify all non-dominated solutions from each subpopulation. Afterwards, the cosine value of the angle between each non-dominated solution and the corresponding reference vector associated with the current subpopulation (e.g., $\vec{v}_k$) is calculated using (6). As a consequence, the individual i, which has the minimal $\theta_{i,k}$, is selected from each subpopulation. In case no individual exists in the corresponding subspace and a subpopulation is empty, the selection procedure will be simply skipped. In this way, since at most one individual is selected from each subpopulation, the size of the entire population $P(t)$ in the next generation is at most K, i.e., $|P(t)| \leq K$.

As a summary of the details above, the pseudo code of the proposed RVPA is presented in Algorithm 1.

Algorithm 1 The pseudo code of the proposed RVPA.

Input: a set of approximate Pareto optimal solutions P^*;
/*initialization*/
Initialization: set $t = 0$, generate a set of reference vectors $\vec{v}_1, ..., \vec{v}_K$ based on the observations on P^*, mix up P^* with a number of K randomized individuals to generate the initial population $P(0)$;
/*main loop*/
while *termination condition* is not satisfied **do**
  Reproduction: generate offspring $Q(t)$ using SBX crossover and polynomial mutation on $P(t)$, merge $Q(t)$ into $P(t)$;
  Reference Vector based Population Division: calculate the cosine values of the acute angles between each individual and each reference vector, and create subpopulations $P_1(t), ..., P_K(t)$ by associating each individual with its closest reference vector;
  for k = 1 to K **do**
    if $P_i(t) \neq \emptyset$ **then**
      Reference Vector based Selection – Step one: do non-dominated sorting on subpopulation $P_k(t)$, and store the first non-dominated front in $F_k^1(t)$;
      Reference Vector based Selection – Step two: on $F_k^1(t)$, find the individual $I_k^*(t)$ which has the smallest acute angle with reference vector $\vec{v}_i$;
      $P(t+1) = P(t+1) \cup I_k^*(t)$;
    end if
  end for
  $t = t + 1$;
end while
Output: preferred Pareto optimal solutions recorded in the final population $P(t)$

IV. Experimental results

To assess the performance of the proposed RVPA, empirical experiments are carried out. Our experiments consist of two parts. In the first part, the preference articulation ability of RVPA is evaluated, and in the second part, the ability of RVPA to improve the quality of existing solutions is evaluated.

TABLE I
The parameter settings of NSGA-II, MOEA/D and RVPA.

Algorithm	Parameter settings
NSGA-II	$p_c = 1.0, \eta_c = 20, p_m = 1/n, \eta_m = 20$
MOEA/D	$T = 20, p_c = 1.0, \eta_c = 20, p_m = 1/n, \eta_m = 20$
RVPA	$p_c = 1.0, \eta_c = 20, p_m = 1/n, \eta_m = 20$

Six three-objective test functions (DTLZ1 to DTLZ6) are selected from the DTLZ suite [18] as benchmark MOPs in the experiments. Since the proposed RVPA requires a set of approximate Pareto optimal solutions as input, two popular MOEAs, the NSGA-II [19] and the MOEA/D [20] are used to obtain the approximate Pareto optimal solutions. The parameter settings of NSGA-II, MOEA/D and the proposed RVPA are listed in Table I, where p_c is the crossover probability, η_c is the crossover index, p_m the mutation probability, η_m the mutation index, n the number of decision variables, and T the neighborhood size.

To measure the quality of the Pareto optimal solutions obtained by different methods, the inverted generational distance (IGD) [21] and the hypervolume (HV) [22] are used as two performance indicators. The two indicators are shown to be able to measure both diversity and convergence of a given set of approximate Pareto optimal solutions (denoted as P^*).

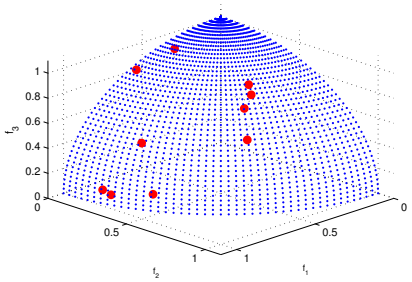
To use the IGD measure, a set of uniformly distributed solutions on the true PF (denoted as P^*) are required, and the IGD between P^* and P can be defined as:

$$IGD(P^*, P) = \frac{\sum_{v \in P^*} d(v, P)}{|P^*|}, \tag{7}$$

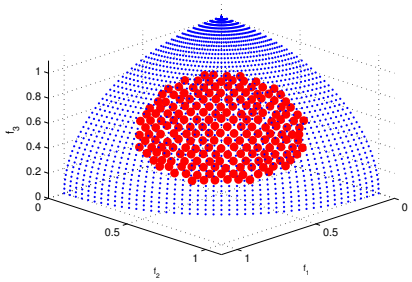
where $d(v, P)$ is the minimum Euclidean distance from point v to P. In this work, 1000 uniformly distributed points are selected from the true PF of each benchmark MOP to be P^*.

To use the HV measure, a reference point (denoted as $y^* = (y_1^*, ..., y_m^*)$) in the objective space which is dominated by all Pareto optimal solutions should be predefined. The HV value of P (with regard to y^*) is the volume of the region which is dominated by P and dominates y^*. In this work, the reference point $y^* = (1.2, 1.2, 1.2)$ is used for all the benchmark MOPs.

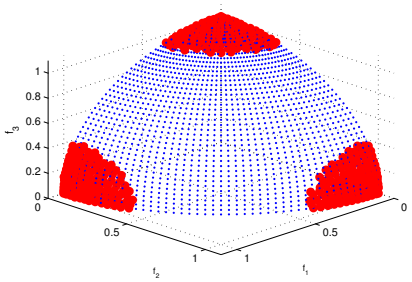
The statistical results presented in this section are all obtained via 20 independent runs, and to compare the results between different methods, the Wilcoxon rank sum test is car-



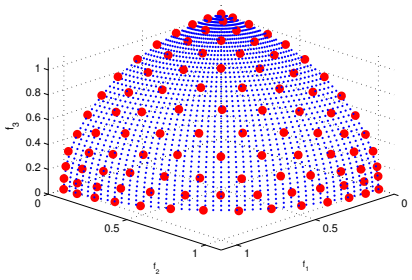
(a) 10 solutions originally obtained by NSGA-II.



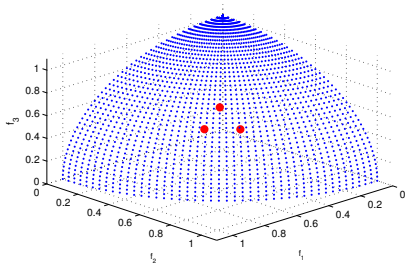
(b) 198 solutions obtained by RVPA with reference vectors set in the middle.



(c) 375 solutions obtained by RVPA with reference vectors set in the corners.



(d) 105 solutions obtained by RVPA with uniformly distributed reference vectors.



(e) Three solutions obtained by RVPA via only 500 fitness evaluations.

Fig. 5. The non-dominated solutions on DTLZ2 obtained by RVPA with different reference vectors.

ried out at a significance level of 0.05. If one group of results are significantly better than the other, they are highlighted in the tables which summarize the statistical results.

A. *Preference articulation*

The preference articulation ability of the proposed RVPA is assessed from two aspects. On the one hand, it is essential to investigate whether RVPA is able to generate solutions in the preferred regions specified by the reference vectors; on the other hand, it is also interesting to know whether RVPA is still able to work properly even if the input approximate Pareto optimal solutions are very sparse. To this end, NSGA-II is run for 100 generations with a population size of 100 to obtain a set of approximate Pareto optimal solutions, and as the input for RVPA, only 10 out of the 100 solutions are randomly selected, as shown in Fig. 5(a).

Firstly, reference vectors are defined in the middle of the objective space. Specifically, the center vector is set as $\vec{c} = (1, 1, 1)$ and the threshold angle θ_r is set to $\frac{\pi}{8}$, which are two parameters used to define the preferred regions, as presented in (4) and (5). To generate reference vectors in the preferred region, a number of 1035 uniformly distributed reference vectors are generated in the entire objective space first, and the reference vectors inside the preferred regions are then picked out. As a result, a total number of 198 reference vectors in the preferred regions are obtained. It turns out that the 198 solutions which corresponds to the 198 reference vectors obtained by RVPA show even distribution in the preferred region (in the middle of the objective space), as evident in Fig. 5(b), which implies that the selection strategy we adopt has successfully guided the population towards the true PF as well as each reference vector.

Similarly, reference vectors defined in the corners of the objective space have also been tested. Correspondingly, the center vectors are set as $\vec{c} = (0, 0, 1)$, $\vec{c} = (0, 1, 0)$ and $\vec{c} = (1, 0, 0)$. The threshold angle θ_r is still $\frac{\pi}{8}$. As a result, a total number of 375 reference vectors are selected from the 1035 uniformly distributed reference vectors, 125 in each corner of the objective space. As shown in Fig. 5(c), the 375 solutions obtained by RVPA show good convergence and distribution in the preferred regions defined by the reference vectors.

In order to visually evaluate the distribution of solutions obtained by RVPA we have also uniformly distributed 105 reference vectors in the entire objective space. As evident in Fig. 5(d), RVPA is able to generate solutions with excellent distribution even if the solutions are spreading relatively sparsely in the objective space.

To obtain the results as shown above, the proposed RVPA has been run for 10000 fitness evaluations (FEs) with each set of reference vectors, respectively, which is a typical number of FEs required by traditional MOEAs. However, since the FE in some practical engineering problems can be fairly expensive [23], [24], such a large number of FEs can be unaffordable. In this case, a decision maker may only set a small number (e.g., two or three) of preferred points instead

of a preferred region to aid the decision making. In order to test whether the proposed RVPA is still able to work properly with a very small population size (as well as a small number of FEs), an additional test has been conducted. In such a test, we set only three reference vectors in the middle of the objective space, and RVPA is run for only 500 FEs, which is much smaller than the normal number of FEs that a traditional MOEA requires to work properly. As shown in Fig. 5(e), RVPA is still to obtain the preferred solutions in such an extreme case.

B. Improving solution quality a posteriori

Since an initial Pareto set of solutions has to be generated a priori we expect, that the quality of the Pareto optimal solutions approximated by a MOEA might be very low due to the fact that the decision maker would like to spend only a minimal number of FEs in the initial step in order to speed up the process and targets to obtain solutions of high quality *a posteriori* in the preferred regions. Therefore, it is of interest to examine whether the proposed RVPA is able to improve the general quality of a given set of approximate Pareto optimal solutions. The comparisons are made between the results obtained by NSGA-II, MOEA/D and RVPA. To start with, we use NSGA-II and MOEA/D to obtain sets of approximate Pareto optimal solutions on DTLZ1 to DTLZ6. Then, on each set of approximate Pareto optimal solutions, RVPA is applied. Since RVPA is operated on the approximate Pareto optimal solutions obtained by NSGA-II and MOEA/D, pairwise comparisons are made between the results obtained by NSGA-II, MOEA/D, and RVPA, respectively.

In order to make the results obtained by NSGA-II, MOEA/D and RVPA comparable, the same population size is used for each algorithm. Since the population size setting of MOEA/D is relevant to the number of uniform weights used, we use a population size of 105 for MOEA/D as well as the other three algorithms. The termination condition of each algorithm is the maximal number of fitness evaluations (FEs). The maximal number of FEs for NSGA-II and MOEA/D is set to 50,000 for DTLZ1, DTLZ3 and 10,000 for the other four functions. For RVPA, we use only 30% the number of FEs as for NSGA-II and MOEA/D, i.e., 15,000 and 3,000, to see whether RVPA is able to significantly improve the solution qualities at a low cost.

The statistical results of the IGD values and HV values obtained by each algorithm are summarized in Table II and Table III. It can be seen that, on all the test functions, RVPA has significantly improved the quality of the solutions obtained by NSGA-II and MOEA/D. For better observations, we have plotted the PFs of the results obtained in some typical runs by each algorithm in Fig. 6. From Fig. 6(a), Fig. 6(b), Fig. 6(i) and Fig. 6(j), it can be observed that RVPA is able to improve the convergence quality of the solutions obtained by NSGA-II on DTLZ1 and DTLZ3, which are two difficult multi-modal functions. From Fig. 6(g), Fig. 6(h), Fig. 6(k) and Fig. 6(i), it can be seen that, although the solutions obtained by MOEA/D on DTLZ2 and DTLZ3 are already of good distributions, RVPA is still able to improve the distribution quality further. Fig. 6(o) and Fig. 6(p) show that even if the majority of the Pareto optimal solutions are missing, RVPA is able to generate supplementary solutions to cover the entire PF. Fig. 6(w) and Fig. 6(x) show that although the PF is not spreading all over the objective space, RVPA is still able to improve the distribution quality as well as the convergence quality of the solutions, as long as the reference vectors are set properly.

TABLE II
STATISTICAL RESULTS OF IGD VALUES OBTAINED BY NSGA-II, MOEA/D AND RVPA. THE SECOND COLUMN AND THE FOURTH COLUMN ARE THE RESULTS OBTAINED BY RVPA BASED ON THE PARETO OPTIMAL SOLUTIONS APPROXIMATED BY NSGA-II AND MOEA/D, RESPECTIVELY.

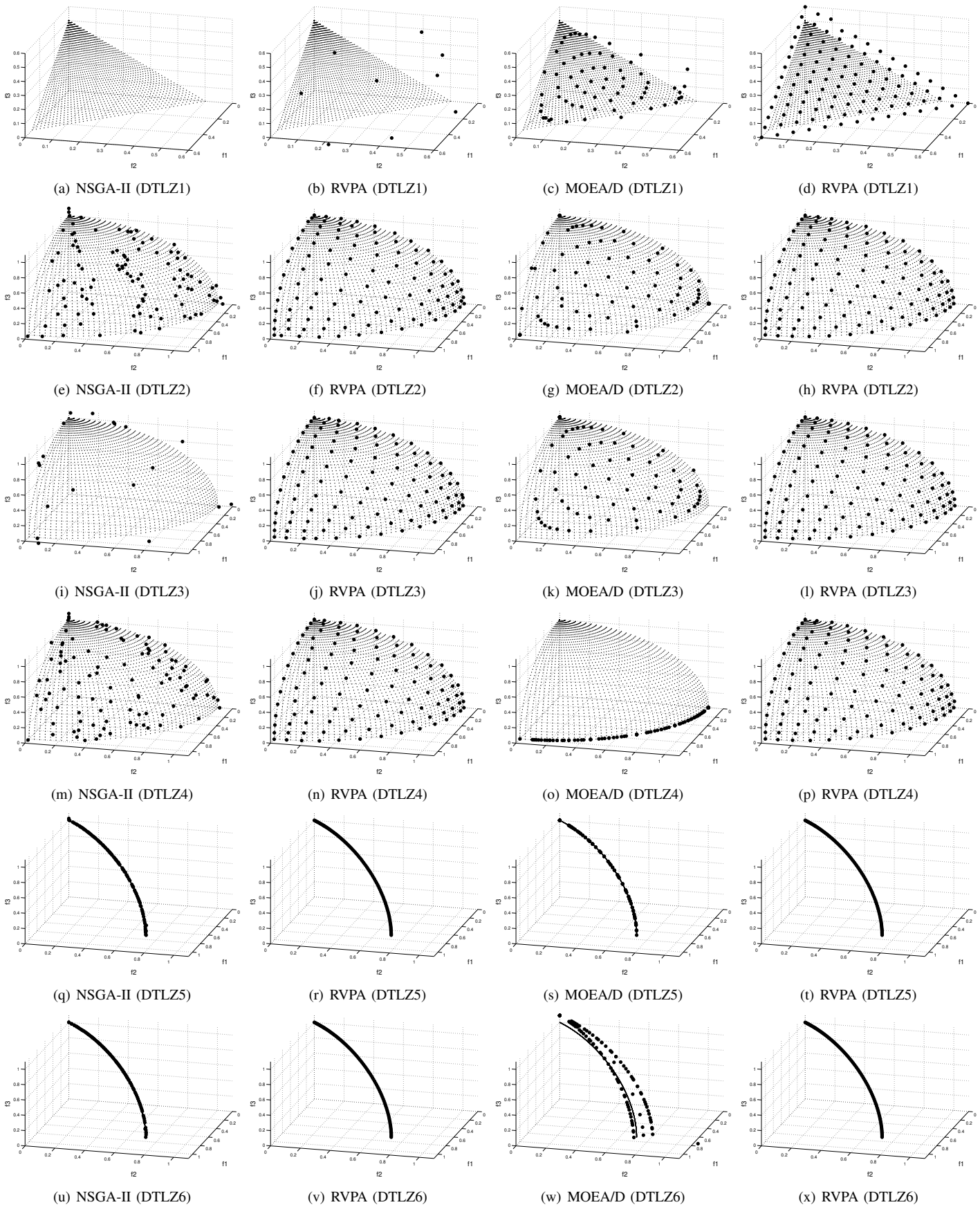
Function	NSGA-II	RVPA	MOEA/D	RVPA
DTLZ1	6.66E+00	**2.17E+00**	7.49E-02	**6.21E-02**
	4.26E+00	**2.77E+00**	1.91E-02	**1.87E-02**
DTLZ2	7.00E-02	**5.08E-02**	6.65E-02	**4.94E-02**
	2.45E-03	**4.48E-04**	5.34E-04	**1.64E-04**
DTLZ3	6.34E-01	**5.73E-02**	7.13E-02	**5.00E-02**
	1.58E+00	**2.08E-02**	3.74E-03	**8.10E-04**
DTLZ4	6.65E-02	**5.09E-02**	3.47E-01	**4.97E-02**
	3.10E-03	**4.35E-04**	3.11E-01	**2.60E-04**
DTLZ5	7.29E-03	**4.44E-03**	1.19E-02	**4.28E-03**
	7.80E-04	**5.16E-05**	9.80E-05	**1.43E-05**
DTLZ6	6.51E-03	**4.31E-03**	1.30E-01	**8.06E-03**
	8.06E-04	**3.16E-05**	2.64E-02	**8.04E-03**

TABLE III
STATISTICAL RESULTS OF HV VALUES OBTAINED BY NSGA-II, MOEA/D AND RVPA. THE SECOND COLUMN AND THE FOURTH COLUMN ARE THE RESULTS OBTAINED BY RVPA BASED ON THE PARETO OPTIMAL SOLUTIONS APPROXIMATED BY NSGA-II AND MOEA/D, RESPECTIVELY.

Function	NSGA-II	RVPA	MOEA/D	RVPA
DTLZ1	6.39E-05	**3.31E-01**	1.58E+00	**1.68E+00**
	2.02E-04	**4.99E-01**	2.11E-02	**8.07E-03**
DTLZ2	1.08E+00	**1.13E+00**	1.10E+00	**1.14E+00**
	1.27E-02	**2.88E-03**	2.81E-03	**9.20E-04**
DTLZ3	8.62E-01	**1.11E+00**	1.01E+00	**1.14E+00**
	3.59E-01	**5.55E-02**	1.60E-02	**5.39E-03**
DTLZ4	1.09E+00	**1.13E+00**	9.43E-01	**1.14E+00**
	7.85E-03	**2.27E-03**	1.78E-01	**1.33E-03**
DTLZ5	7.19E-01	**7.23E-01**	7.17E-01	**7.23E-01**
	9.01E-04	**1.83E-04**	3.94E-04	**5.56E-05**
DTLZ6	7.19E-01	**7.23E-01**	5.53E-01	**7.16E-01**
	1.27E-03	**2.83E-04**	3.50E-02	**1.61E-02**

V. CONCLUSION

This paper has proposed a reference vector based preference articulation (RVPA) method for evolutionary multiobjective optimization. The proposed RVPA is able to obtain solutions in the preferred regions (or points) specified by the reference vectors defined in the objective space. Experimental results show that the proposed RVPA is not only able to deal with various kinds of preferences effectively and efficiently, but also able to improve the quality of a given set of approximate Pareto optimal solutions obtained by an MOEA *a posteriori*. In addition, an attractive feature of the proposed RVPA is that it is able to work with a very small population size as well as a very limited number of fitness evaluations, which implies promising applicability of RVPA on real world engineering problems whose fitness evaluations can be very expensive.



(a) NSGA-II (DTLZ1) (b) RVPA (DTLZ1) (c) MOEA/D (DTLZ1) (d) RVPA (DTLZ1)

(e) NSGA-II (DTLZ2) (f) RVPA (DTLZ2) (g) MOEA/D (DTLZ2) (h) RVPA (DTLZ2)

(i) NSGA-II (DTLZ3) (j) RVPA (DTLZ3) (k) MOEA/D (DTLZ3) (l) RVPA (DTLZ3)

(m) NSGA-II (DTLZ4) (n) RVPA (DTLZ4) (o) MOEA/D (DTLZ4) (p) RVPA (DTLZ4)

(q) NSGA-II (DTLZ5) (r) RVPA (DTLZ5) (s) MOEA/D (DTLZ5) (t) RVPA (DTLZ5)

(u) NSGA-II (DTLZ6) (v) RVPA (DTLZ6) (w) MOEA/D (DTLZ6) (x) RVPA (DTLZ6)

Fig. 6. The PFs approximated by NSGA-II, MOEA/D and RPVA on DTLZ1 to DTLZ6 (on each row). The second column and the fourth column are the results obtained by RVPA based on the Pareto optimal solutions approximated by NSGA-II and MOEA/D, respectively.

The current RVPA only works based on a given set of approximate Pareto optimal solutions obtained by an existing MOEA. In the future, it can be interesting to investigate whether such a framework is able to work independently as an MOEA after some modifications. In addition, the performance of the proposed RVPA on many-objective problems (where the number of objectives is larger than three) also remains to be examined.

ACKNOWLEDGEMENT

This work was supported in part by Honda Research Institute Europe and the Joint Research Fund for Overseas Chinese, Hong Kong and Macao Scholars of the National Natural Science Foundation of China (Grant No. 61428302).

REFERENCES

[1] Y. Jin and B. Sendhoff, "Connectedness, regularity and the success of local search in evolutionary multi-objective optimization," in *Proceedings of the IEEE Congress on Evolutionary Computation*, vol. 3. IEEE, 2003, pp. 1910–1917.

[2] Q. Zhang, A. Zhou, and Y. Jin, "RM-MEDA: A regularity model-based multiobjective estimation of distribution algorithm," *IEEE Transactions on Evolutionary Computation*, vol. 12, no. 1, pp. 41–63, 2008.

[3] A. Zhou, B.-Y. Qu, H. Li, S.-Z. Zhao, P. N. Suganthan, and Q. Zhang, "Multiobjective evolutionary algorithms: A survey of the state of the art," *Swarm and Evolutionary Computation*, vol. 1, no. 1, pp. 32–49, 2011.

[4] C. A. Coello Coello, "Handling preferences in evolutionary multiobjective optimization: A survey," in *Proceedings of the Congress on Evolutionary Computation*, vol. 1. IEEE, 2000, pp. 30–37.

[5] Y. Jin and B. Sendhoff, "Incorporation of fuzzy preferences into evolutionary multiobjective optimization," in *Proceedings of the Genetic and Evolutionary Computation Conference*. ACM, 2002, pp. 26–30.

[6] R. T. Marler and J. S. Arora, "The weighted sum method for multi-objective optimization: new insights," *Structural and multidisciplinary optimization*, vol. 41, no. 6, pp. 853–862, 2010.

[7] K. Miettinen, F. Ruiz, and A. P. Wierzbicki, "Introduction to multiobjective optimization: interactive approaches," in *Multiobjective Optimization*. Springer, 2008, pp. 27–57.

[8] K. Miettinen, *Nonlinear multiobjective optimization*. Springer, 1999, vol. 12.

[9] I. Giagkiozis and P. J. Fleming, "Pareto front estimation for decision making," *Evolutionary Computation*, accepted in 2014.

[10] K. Deb, J. Sundar, N. Udaya Bhaskara Rao, and S. Chaudhuri, "Reference point based multi-objective optimization using evolutionary algorithms," *International Journal of Computational Intelligence Research*, vol. 2, no. 3, pp. 273–286, 2006.

[11] J. Molina, L. V. Santana, A. G. Hernández-Díaz, C. A. C. Coello, and R. Caballero, "g-dominance: reference point based dominance for multiobjective metaheuristics," *European Journal of Operational Research*, vol. 197, no. 2, pp. 685–692, 2009.

[12] L. Thiele, K. Miettinen, P. J. Korhonen, and J. Molina, "A preference-based evolutionary algorithm for multi-objective optimization," *Evolutionary Computation*, vol. 17, no. 3, pp. 411–436, 2009.

[13] H.-L. Liu, F. Gu, and Q. Zhang, "Decomposition of a multiobjective optimization problem into a number of simple multiobjective subproblems," *IEEE Transactions on Evolutionary Computation*, vol. 18, no. 3, pp. 2450–455, 2014.

[14] K. Li, Q. Zhang, and S. Kwong, "An evolutionary many-objective optimization algorithm based on dominance and decomposition," *IEEE Transactions on Evolutionary Computation*, accepted in 2014.

[15] R. Cheng, Y. Jin, K. Narukawa, and B. Sendhoff, "A multiobjective evoltuionary algorithm using Gaussian process based inverse modeling," *IEEE Transactions on Evolutionary Computation*, accepted in 2015.

[16] R. Cheng, Y. Jin, and K. Narukawa, "Adaptive reference vector generation for inverse model based evolutionary multiobjective optimization with degenerate and disconnected pareto fronts," in *Evolutionary Multi-criterion Optimization*. Springer, 2015.

[17] J. A. Cornell, *Experiments with mixtures: designs, models, and the analysis of mixture data*. John Wiley & Sons, 2011.

[18] K. Deb, L. Thiele, M. Laumanns, and E. Zitzler, "Scalable multi-objective optimization test problems," in *Proceedings of the Congress on Evolutionary Computation*. IEEE, 2002, pp. 825–830.

[19] K. Deb, A. Pratap, S. Agarwal, and T. Meyarivan, "A fast and elitist multiobjective genetic algorithm: NSGA-II," *IEEE Transactions on Evolutionary Computation*, vol. 6, no. 2, pp. 182–197, 2002.

[20] Q. Zhang and H. Li, "MOEA/D: A multiobjective evolutionary algorithm based on decomposition," *IEEE Transactions on Evolutionary Computation*, vol. 11, no. 6, pp. 712–731, 2007.

[21] P. A. Bosman and D. Thierens, "The balance between proximity and diversity in multiobjective evolutionary algorithms," *IEEE Transactions on Evolutionary Computation*, vol. 7, no. 2, pp. 174–188, 2003.

[22] K. Deb *et al.*, *Multi-objective optimization using evolutionary algorithms*. John Wiley & Sons Chichester, 2001, vol. 2012.

[23] Y. Jin, "A comprehensive survey of fitness approximation in evolutionary computation," *Soft Computing*, vol. 9, no. 1, pp. 3–12, 2005.

[24] ——, "Surrogate-assisted evolutionary computation: Recent advances and future challenges," *Swarm and Evolutionary Computation*, vol. 1, no. 2, pp. 61–70, 2011.