Kriging-Assisted Topology Optimization of Crash Structures

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Abstract
Over the recent decades, Topology Optimization (TO) has become an important tool in the design and analysis of mechanical structures. Although structural TO is already used in many industrial applications, it needs much more investigation in the context of vehicle crashworthiness. Indeed, crashworthiness optimization problems present strong nonlinearities and discontinuities, and gradient-based methods are of limited use. The aim of this work is to present an in-depth analysis of the novel Kriging-Assisted Level Set Method (KG-LSM) for TO. It is based on an adaptive optimization strategy using the Kriging surrogate model and a modified version of the Expected Improvement (EI) as the update criterion, which allows for embedding opportune constraints. The adopted representation using Moving Morphable Components (MMCs) significantly reduces the dimensionality of the problem, enabling an efficient use of surrogate-based optimization techniques. A minimum compliance cantilever beam test case of different dimensionalities is used to validate the presented strategy, as well as identify its potential and limits. The method is then applied to a 2D crash test case, involving a cylindrical pole impact on a rectangular beam fixed at both ends. Compared to the state-of-the-art Covariance Matrix Adaptation Evolution Strategy (CMA-ES), the KG-LSM optimization algorithm demonstrates to be efficient in terms of convergence speed and performance of the optimized designs.

Keywords: Topology Optimization, Crashworthiness Optimization, Surrogate Modeling, Kriging, Level Set Method, Moving Morphable Components

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Nomenclature

\begin{itemize}
\item \( D \) design domain
\item \( \Omega \) region occupied by material
\item \( u \) point in the design domain
\item \( \Phi \) global level set function
\item \( \phi_i \) local level set function
\item \( e \) number of elementary components
\item \( x_0 \) \( x \)-coordinate of the component barycenter
\item \( y_0 \) \( y \)-coordinate of the component barycenter
\item \( l \) length of the elementary component
\item \( t \) thickness of the elementary component
\item \( \theta \) rotation angle of the elementary component
\item \( \rho \) density function
\item \( E^0 \) reference stiffness tensor
\item \( H \) Heaviside function
\item \( f_{\text{obj}} \) objective function
\item \( g_i \) \( i \)th inequality constraint
\item \( x \) vector of design variables
\item \( X \) training set
\item \( y \) vector of observed responses
\item \( Y \) random vector of responses
\item \( n \) number of sample points
\item \( k \) dimensionality of the problem
\item \( \theta \) vector of widths for Kriging basis functions
\item \( p \) smoothness parameter of Kriging basis functions
\item \( \mu \) mean of the responses random vector
\item \( \sigma \) variance of the responses random vector
\item \( \hat{\mu} \) maximum likelihood estimate of mean
\item \( \hat{\sigma} \) maximum likelihood estimate of the variance
\item \( \Psi \) correlation matrix between the random variables
\item \( \tilde{y} \) augmented response vector
\item \( \tilde{\Psi} \) augmented correlation matrix
\item \( \hat{y} \) prediction of the response at a new location
\item \( \hat{s}^2 \) prediction of the committed error in the prediction
\item \( x^* \) best design variable
\item \( y^* \) best response
\item \( \Phi_Y \) Gaussian cumulative distribution function
\item \( \varphi_Y \) probability density function
\end{itemize}
1. Introduction

A lot of factors, such as increased traffic intensity, manufacturers’ desire to produce always better performing cars, growing concern of the general public, and more stringent regulations, have made vehicle crashworthiness one of the major research areas in automotive engineering. As in other industrial applications, iterative physical experiments are characterized by prohibitive times and costs, and substantial effort has to be put into the project and design phase of the new products. In particular, due to the development of Computer-Aided Design (CAD) methods and the progress in numerical simulations with the Finite Element Method (FEM), together with the recent advancements in innovative manufacturing technologies, increasing attention has been put in the prototyping of mechanical structures, where structural Topology Optimization (TO) \cite{1} has demonstrated to be particularly useful. First introduced by Bendsøe and Kikuchi \cite{2}, TO is a powerful approach, which addresses the task of determining optimal concept structures through changing the material distribution in a given design domain. It has proven to be a valuable tool in early phases of the design process and as such it gained popularity in lightweight design of structures in automotive and aerospace industry, as well as in civil engineering, materials science, and biomechanics. Most of the currently used structural TO methods are density-based approaches, which fall into two general categories: the SIMP-based methods \cite{3, 4} and the homogenization methods \cite{2, 5}. These strategies are based on the parametrization of the design by splitting the domain into a mesh of elements whose densities change within the optimization process, leading to modifications in the material distribution. Level Set Methods (LSMs) represent an alternative to density-based approaches in structural TO. The concept of LSMs was first introduced by Osher and Sethian \cite{6} and successively extended to the geometry descriptions in TO by Haber et al. \cite{7}. From then on, Level Set-based TO \cite{8, 11} received the attention of many researchers for its ability to handle rather large topological changes while keeping a simple analytical representation throughout the optimization process. In fact, rather than by modifying the densities of elements on the discretized domain, the material distribution changes according to an implicit parametrization of the material boundaries by means of Level Set Functions (LSFs).

These properties of level set-based representations allow for the efficient utilization of global optimization methods. The effect is that simplifications and heuristics frequently introduced in density-based methods can be relaxed. Important phenomena like for example material contact and material failure, which result in high nonlinearities, noisiness and discontinuities of the response, can be considered in the optimization process. Therefore, LSF representations allow to overcome major drawback of state of the art methods which will be discussed in the following.

The Equivalent Static Loads method (ESL) was introduced for structural size and shape optimization by Kang et al. \cite{12} and extended to nonlinear static and dynamic responses by Park \cite{13, 14} to address the challenging problem of nonlinearities in vehicle crash events, by defining sets of static loads to roughly represent the dynamic impact conditions. Afterwards, ESL was applied to TO \cite{15, 16}, where efficient gradient-based optimization techniques can be used for finding optimal configurations by assuming the equivalence of static and dynamic loads \cite{17}. However, within this approach, the static loads defining the linear elastic problem are often insufficient to reproduce the dynamic effects on structural components undergoing large deformations. In addition, it is ques-
tionable if linear sensitivities are appropriate to obtain an optimal solution for nonlinear problems.

The Ground Structure Approaches (GSAs) [18] allow for obtaining a simplified crash behavior through the definition of elementary beam elements in the design space. Thanks to the availability of analytical gradients, thicknesses of the components can be adjusted based on gradient-based optimizers. The initial (ground) structure is formed by a large number of elements, which are subsequently deleted if the thickness of a component drops below a given threshold. This process goes on until a satisfying objective target is reached. The first limitation is that GSA can afford a relatively low number of design variables with restricted variability, leading to an optimization that is significantly dependent on the initial ground structure [19]. Moreover, it is difficult to calibrate the simplified model used in GSA to match the crash behavior of the real structure [20]. In order to overcome these difficulties and reduce computational costs, Zhang et al. recently introduced an efficient randomized approach for topology optimization of practical structural systems under many load cases [21]. However, many questions remain still open, highlighting the need for further research.

Bubble and Graph/Heuristic-based approaches [22, 23] were proposed for a combined topology and shape optimization. The solution concept of the Bubble Method comprises iterative positioning of small holes into the structure of the component being optimized, based on either analytical or numerical laws, followed by the shape optimization of the new bubbles together with the boundaries of the component. Since the described optimization procedure carries out a numerical sensitivity analysis, the computation time for the optimization is substantial and the method is not appropriate for optimization of complex structures. On the other hand, Graph/Heuristic-based methods [24] start from a graph representing the studied mechanical structure and divide the problem into two different optimization loops: the outer loop, dealing with the structure’s topology and shape modification, and the inner one, performing size and shape optimization, e.g. with the use of evolutionary algorithms. These algorithms are capable of optimizing structures with respect to crash load cases, but they also require a great amount of computing power, especially during the inner loop. Moreover, the topology modifications are made according to heuristic rules based on the expert knowledge and, as such, are not suitable in general.

The Hybrid Cellular Automata method [25, 26] is a biologically inspired algorithm, based on the evolution of a cellular automaton, consisting of a regular grid of cells, which can take one of the finite number of states according to local rules, as well as global information obtained from the finite element analysis. The objective is to find a material distribution that homogenizes the internal energy density of a structure, which is a working heuristic for stiffness and some crash objectives. Despite the usefulness of the method in the automotive industry thanks to a computational efficiency comparable to gradient-based methods [27], the heuristic nature of the crash optimization is not suitable for modeling vehicle components undergoing large plastic deformations and complex interactions during the crash event, e.g. buckling or failure behavior. To overcome this limit, the Hybrid Cellular Automata for Thin-Walled Structures (HCATWS) has been proposed [28, 30]. In spite of the low computational costs associated with this optimization method, it is fundamentally based on the heuristic assumption of energy homogenization, which limits its generalization.

State-based representation approaches [31, 32] address the task of reducing the di-
mensionality of the optimization problem by assuming that density changes of elements having the same local state features (e.g., energies or displacements) also influence the objective function in a similar way. At first, the elements filling the design space are clustered according to the local state features. Afterwards, an evolutionary optimization generates optimizing update-signal parameters and the derived sensitivity model is applied in a gradient-based algorithm. The main drawback of this method is the requirement of computationally expensive retraining and more applications to assess its potential are needed.

The Evolutionary Level Set Method (EA-LSM) for crash Topology Optimization [33, 34] uses Evolutionary Algorithms (EAs) [35] to carry out the optimization procedure when no reliable gradient information can be used. Indeed, Evolution Strategies (ES) do not require any previous heuristic assumptions and directly rely on evaluations of the objective function. Moreover, if compared to other global optimization techniques, EAs are easy to implement and very often provide adequate solutions, showing many advantages such as simplicity, robust responses to changing circumstances, flexibility, and many other. Since these favorable characteristics result in relatively high computational costs, methods to mitigate this problem by using approximate gradient information [36, 37] and prediction of topology variations [38] were also proposed. The method was also adapted to deal with crash topology optimization of thin-walled extrusions, highly relevant for the automotive industry [39]. However, in view of the final application, further work on saving computational effort is needed.

By taking advantage of the low-dimensional parametrization introduced within the EA-LSM, based on Moving Morphable Components, this research proposes a novel optimization strategy introducing Surrogate Modeling Techniques [40], also known as Meta-modeling. Based on a certain number of finite element analyses, Surrogate Models allow for the construction of a computationally cheap-to-evaluate approximation of the considered expensive objective function and replace the direct optimization of the real objective with the model. As a consequence, many more evaluations can be performed on the approximate model, requiring very few high-fidelity FEM evaluations to increase its accuracy in the promising regions of the design space.

In the literature, the most widely used surrogate models are: Polynomial Regression [40, 41], Radial Basis Function [40, 42], Kriging [40, 43, 44], Support Vector Regression [45, 46], Artificial Neural Networks [47, 48] and others. Different models might provide different interpretation of the objective function or slightly different accuracies. Although each surrogate modeling technique has its major advantages and disadvantages and it is unclear in general which model is the most suitable, Kriging models are chosen in this research due to their ability to estimate the potential error committed in the approximation. At each iteration of the optimization algorithm, the Kriging surrogate model is refitted and the parameters characterizing the approximation are estimated. The estimate of the error is a crucial information in the update phase of the optimization procedure, which is carried out in this work through the maximization of opportune variants of the Expected Improvement (EI) function, further described below.

Taking advantage of the potential of the Kriging metamodel, this research presents the Kriging-guided Level Set Topology Optimization Method (KG-LSM) [49, 50], which couples Kriging with the Level Set Method in the Topology Optimization of representative structural problems. In this research, the Kriging surrogate represents an innovation in the LSM update process. In fact, in most of the Level Set approaches, the LSF is
dynamically updated at each time step by solving the Hamilton-Jacobi Partial Differential Equation [51], which describes the motion of the material boundaries according to a certain velocity field. Alternatively, mathematical programming approaches like the Steepest Descent Method (SDM) [52, 53] or the Method of Moving Asymptotes (MMA) [54, 55] can be used. Both of these classes of update techniques cannot be used for particular categories of optimization problems, such as crashworthiness, where no sensitivity information is available.

In order to assess the efficiency of the proposed strategy, the obtained results are compared to the state-of-the-art Covariance Matrix Adaptation Evolution Strategy (CMA-ES) [56]. The methods are evaluated on a static cantilever beam benchmark test case for a fixed budget of FEM evaluations. This choice is dictated by the fact that in many industrial applications, FEM evaluations are very expensive and, as such, limited. The proposed optimization strategy is also applied to a dynamic pole intrusion test case, and the convergence trends, as well as the optimized designs resulting from the considered optimization algorithms, are discussed.

The paper has the following structure. Section 2 introduces the adopted representation of the problem, consisting of an implicit parametrization with geometric LSFs and a density-based geometry mapping. Section 3 analyzes the considered optimization problem and the constraints that have to be taken into account in order to produce well-performing mechanical structures. In Section 4 the adopted optimization strategy, by means of a Kriging-assisted update process, is presented. Moreover, the techniques developed to handle constraints are presented. Afterwards, Section 5 describes the experimental test cases, and the obtained results are discussed in Section 6. Final conclusions are drawn at the end.

2. Problem Representation

The KG-LSM utilizes the Kriging surrogate model in the TO of structures represented by means of the LSM. In this framework, the material boundary is implicitly defined by iso-contours of a LSF.

In Sections 2.1 and 2.2 the parametrization and the geometry mapping to the mechanical model are described, respectively.

2.1. Parametrization

Let a bounded domain \( D \in \mathbb{R}^2 \) be the design domain including all the admissible shapes, and \( \Omega \) be the region of the domain \( D \) occupied by material. The global LSF \( \Phi \) has the following characterization:

\[
\begin{align*}
\Phi(u) > 0, & \quad u \in \Omega, \\
\Phi(u) = 0, & \quad u \in \partial \Omega, \\
\Phi(u) < 0, & \quad u \in D \setminus \Omega,
\end{align*}
\]

(1)

where \( \Omega \) is the region of the domain \( D \) occupied by material, \( D \setminus \Omega \) is its complementary set, occupied by void, and \( \partial \Omega \) is the interface between material and void. Hence, the 0\(^{th}\) iso-contour clearly defines the interface between different phases of the material, avoiding intermediate densities for each \( u = (x, y)^T \in D \).
Since in structural mechanics optimal topologies frequently consist of ensembles of interconnected beams, a Level Set description, inspired by Guo et al. [57], and later applied by Bujny et al. [37], is used. The global LSF is then composed of local LSFs, each representing an elementary beam component able to deform and move in the design domain. Every local LSF is given by:

\[
\begin{cases}
\phi_i(u) > 0, & u \in \Omega_i, \\
\phi_i(u) = 0, & u \in \partial \Omega_i, \\
\phi_i(u) < 0, & u \in D \setminus \Omega_i,
\end{cases}
\]  

(2)

where \( \phi_i \) is the LSF which parametrizes the \( i^{th} \) component that occupies the region \( \Omega_i \) of design domain.

Therefore, if \( e \) is the number of elementary components, the region of the domain occupied by material is identified as:

\[
\Omega = \bigcup_{i=1}^{e} \Omega_i.
\]  

(3)

For \( D = \mathbb{R}^2 \), every beam component can be represented by a local LSF of the following form:

\[
\phi_i(u) = -\left[ \left( \frac{\cos \theta_i(x-x_0_i)+\sin \theta_i(y-y_0_i)}{l_i/2} \right)^m + \left( \frac{-\sin \theta_i(x-x_0_i)+\cos \theta_i(y-y_0_i)}{t_i/2} \right)^m - 1 \right],
\]  

(4)

where \((x_0, y_0)\) denotes the position of the center of the component with length \( l \), thickness \( t \), and oriented inside the domain according to a rotation angle \( \theta \), as shown in Figure 1a. In Equation (4), the parameter \( m \) is a relatively large even integer number, equal to 6 in this work, as defined in the other papers [37, 57]. Therefore, each beam is described by a vector of five design variables, \((x_0, y_0, l, t, \theta)\), and by changing the values of the parameters, each beam can move, dilate or shrink and rotate in the design domain. Figure 1b shows a 3-dimensional plot of the described local LSF.

Figure 1: Structural components details [37]: (a) Component parametrization, (b) Corresponding local LSF, where negative values are set to zero.
2.2. Geometry Mapping

In Level Set TO it is essential to choose an appropriate mapping between the geometry defined by the LSF and the mechanical model. Here, a density-based geometry mapping is used:

\[ E(u) = \rho(u)E^0 \quad 0 \leq \rho(u) \leq 1, \quad (5) \]

where \( E^0 \) is the reference value of the stiffness tensor and \( \rho(u) \) is the density at the point \( u \in D \).

In turn, the relation between the density \( \rho(u) \) at position \( u \) and the LSF \( \Phi(u) \) is given by:

\[ \rho(u) = H(\Phi(u)), \quad (6) \]

where the global LSF can be represented as:

\[ \Phi(u) = \max(\phi_1(u), \phi_2(u), ..., \phi_e(u)), \quad (7) \]

and \( H(x) \) is the Heaviside function, which assumes 0 value for negative arguments and 1 for nonnegative ones. An example of a composition of local LSFs aggregated using Equation (7) is illustrated in Figure 2.

![Figure 2](image)

(a) Combination of local Level Set Functions: (a) structural layout, (b) corresponding global LSF, where negative values are set to zero.

3. Optimization Problem and Constraints

In this paper, an optimization problem of the following form is considered:

\[
\begin{align*}
\min_x \quad & f_{\text{obj}}(x), \\
\text{s.t.} \quad & r(t) = 0, \\
& g_i(x) \leq 0, \quad i = 1, ..., m, \quad x \in \mathbb{R}^k,
\end{align*}
\]

where \( f_{\text{obj}} \) is the objective function to be minimized and \( g_i, \ i = 1, ..., m, \) are the inequality constraints. The objective function is computed on the vector \( x \in \mathbb{R}^k \) of the design variables, which collects all the parameters defining the LSM basis functions, and \( r(t) = 0 \) expresses the equilibrium condition at time \( t \). In Section 5.1 the objective
function $f_{\text{obj}}$ is the compliance of a cantilever beam. In Section 6.2, pole intrusion is considered as the objective function. In both test cases, the optimization is carried out while respecting a $g(x) = V(x) - V_{\text{req}} \leq 0$ constraint, which requires the volume $V$ of the analyzed structure not to exceed a prescribed limit $V_{\text{req}}$, corresponding to 50% of the design domain volume. It is important to underline that compliance and intrusion minimization are just representatives of a much larger basin of optimization problems, and the proposed methodology can deal with almost any other quantifiable objective functions and constraints. For instance, another static optimization problem might consist of volume minimization with compliance constraint, while mass minimization with maximal intrusion and acceleration constraints might be addressed as a dynamic test.

As a result of the Design of Experiments (DoE) phase or after the infill procedure, disconnected structures (Figure 3), which are not suitable for the task they are designed to deal with, can be obtained. In order to be physically consistent, an optimal design must satisfy appropriate connectivity conditions. Therefore, a connectivity constraint is also considered within this work. While the volume constraint takes into account the industrial requirements of limited mass of the concept design, the connectivity constraint ensures that the obtained designs make sense from the physical point of view. Each material distribution has to be connected to the supports, to eventual loads, and has to respect a connection of the structure to itself, i.e. a material path from the support to the load (cantilever beam test case), or between the two supports (pole intrusion test case) has to be found. Figure 3 shows three material distributions of the cantilever beam test case, each representing the violation of the connectivity constraint according to the first, second, and third criterion, respectively. Several techniques have been proposed to handle such constraints. They are described in Section 4.2.

4. Resolution Strategy

The following section focuses on the description of the optimization procedure defined within this research. Since the investigated optimization problem is characterized by strong nonlinearities, numerical noise and severe discontinuities in the objective function, an alternative approach to the ones relying on analytical sensitivities or heuristic assumptions is presented. By embedding a Kriging metamodel, the proposed optimization strategy drives the material distribution on the design domain towards an optimal configuration. Below, the optimization algorithm (Section 4.1) and techniques to handle the required constraints (Section 4.2) are introduced.
4.1. Optimization Algorithm

The update procedure proposed in this work is tailored to solve efficiently problems in structural mechanics. It is a redraft of the Efficient Global Optimization (EGO) algorithm [44, 59], applied to the Kriging surrogate model. Algorithm 1 gives an outline of the developed optimization procedure in case if each constraint is required to be fulfilled.

**Data:** initial data set \((X, Y)\) with \(n\) sample points, where
\[X = \{x^{(1)}, x^{(2)}, \ldots, x^{(n)}\}^T\]

\[i := 1;\]

**while** \(i < n\) **do**

  **check feasibility of point** \(x^{(i)}\);
  **if** \(x^{(i)}\) **infeasible** **then**

    \[(X, Y) := (X, Y)\backslash \{x^{(i)}, y^{(i)}\};\]

  **end**

  \[i := i + 1;\]

**end**

\[t := 0;\]

**while** \(t < n_{\text{max}}\) **do**

  **fit Kriging to available data** \((X, Y)\);
  **find infill point:** \(p := \arg \max_x \text{EICD}(x)\) (Equation (22));
  **update sample set:** \(X := X \cup p;\)
  **update response set:** \(Y := Y \cup y(p);\)

  \[t := t + 1;\]

**end**

\[y^* := \min(Y);\]

\[x^* := x \in X : y(x) = y^*.\]

**Algorithm 1:** KG-LSM optimization algorithm.

The EGO algorithm starts with the DoE [40, 60], which samples the design space and selects a set of training points where to evaluate the high-fidelity model. In order to determine a set of training samples, an Optimal Latin Hypercube Sampling (OLHS) [61] is used in this work. The training data \(X = \{x^{(1)}, x^{(2)}, \ldots, x^{(n)}\}^T\) with observed responses \(y = \{y^{(1)}, y^{(2)}, \ldots, y^{(n)}\}^T\), are then used by the Kriging model to predict the value of the objective function at each location \(x^{(n+1)} \in D\). The approximation procedure works as follows.

The DoE data are seen as results of a stochastic process, which is described by using a set of random vectors \(Y(x) = [Y(x^{(1)}), \ldots, Y(x^{(n)})]^T\), with mean \(1\mu\), where \(1\) is an \(n \times 1\) column vector of ones. The correlation between each couple of random variables, \(x^{(i)}\) and \(x^{(l)}\), is described using a squared-exponential basis function expression:

\[
\text{cor}[Y(x^{(i)}), Y(x^{(l)})] = \exp \left( - \sum_{j=1}^{k} \theta_j |x^{(i)}_j - x^{(l)}_j|^p \right),
\] (9)
where $k$ is the dimensionality of the problem, the $\theta$ vector represents the width of each basis function, while $p$ is taken equal to 2 and controls the smoothness of the approximation in the proximity of the given sample points. By definition, the correlations depend on the absolute distance between the sample points as well as the parameters $\theta_j$, which are estimated as follows. This task is fulfilled by maximizing the likelihood function of the predicted data $y$, defined as:

$$L(Y^{(1)}, ..., Y^{(n)}|\mu, \sigma) = \frac{1}{(2\pi\sigma^2)^{n/2}}\exp\left(-\frac{1}{2\sigma^2}\sum(Y^{(i)} - \mu)^2\right)$$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}|\Psi|^{1/2}}\exp\left(-\frac{(y - 1\mu)^T\Psi^{-1}(y - 1\mu)}{2\sigma^2}\right),$$  

where $\Psi$ is the correlation matrix between the random variables, defined as:

$$\Psi = \begin{pmatrix}
\text{cor}[Y(x^{(1)}), Y(x^{(1)})] & \cdots & \text{cor}[Y(x^{(1)}), Y(x^{(n)})] \\
\vdots & \ddots & \vdots \\
\text{cor}[Y(x^{(n)}), Y(x^{(1)})] & \cdots & \text{cor}[Y(x^{(n)}), Y(x^{(n)})]
\end{pmatrix}. $$ (11)

The natural logarithm of Equation (10) is then considered. By setting its derivatives to zero, the maximum likelihood estimates (MLEs) for the mean $\mu$ and variance $\sigma^2$ are obtained:

$$\hat{\mu} = \frac{1^T\Psi^{-1}y}{1^T\Psi^{-1}1}, \quad \hat{\sigma}^2 = \frac{(y - 1\mu)^T\Psi^{-1}(y - 1\mu)}{n}. $$ (12)

As shown by Equations (11) and (10), such estimates depend on the parameters $\theta_j$. Therefore, $\hat{\mu}$ and $\hat{\sigma}^2$ can be substituted into Equation (10), and, since the problem data is known, the numerical maximization of the likelihood function gives back the estimation of the concerned parameters.

Finally, once the correlation parameters that maximize the likelihood of the observed data are found, the model correlation can be used to predict new values based on the observed data. Each new prediction $\hat{y}$ should be consistent with the observed data and the correlation parameters, and it should hence maximize the likelihood of the sample data together with the prediction itself. To achieve this, the model data is augmented with a new input $x$ and the corresponding output $\hat{y}$, the values of which have to be determined:

$$\tilde{y} = (y^T, \hat{y})^T \quad \text{and} \quad \tilde{\Psi} = \begin{pmatrix}
\Psi \\
\psi^T \\
1
\end{pmatrix}, $$ (13)

where $\psi$ is the vector of correlations between the observed data and the sought prediction:

$$\psi = \begin{pmatrix}
\text{cor}[Y(x^{(1)}), Y(x)] \\
\vdots \\
\text{cor}[Y(x^{(n)}), Y(x)]
\end{pmatrix}. $$ (14)

Now, by maximizing the likelihood of the augmented data, the prediction of the response at the new location $\hat{y}$ and the committed error in the prediction $\hat{s}^2$ are determined:

$$\hat{y}(x) = \hat{\mu} + \psi^T\Psi^{-1}(y - 1\mu), \quad \hat{s}^2(\mathbf{x}) = \hat{\sigma}^2 \left[1 - \psi^T\Psi^{-1}\psi + \frac{1 - 1^T\Psi^{-1}\psi}{1^T\Psi^{-1}1}\right]. $$ (15)
After the first approximation model is constructed, an update process is carried out by using the Differential Evolution (DE) \[62, 63\] evolutionary algorithm in the maximization of proposed variants (described in Section 4.2) of the standard Expected Improvement (EI) function \[40\], originally defined as:

\[
E[I(x)] = \begin{cases} 
(y^* - \hat{y}(x)) \Phi_Y \left( \frac{y^* - \hat{s}(x)}{\hat{s}(x)} \right) + \hat{s}(x) \varphi_Y \left( \frac{y^* - \hat{s}(x)}{\hat{s}(x)} \right) & \text{if } \hat{s}(x) > 0, \\
0 & \text{if } \hat{s}(x) = 0.
\end{cases}
\]  

(16)

In Equation (16), \( y^* \) is the best objective value from FEM evaluations so far, while \( \Phi_Y \) and \( \varphi_Y \) are the Gaussian cumulative distribution function and probability density function, respectively. The uncertainty on the interpolated function value, obtained in Equation (15), is utilized to balance between exploitation and exploration. In fact, Equation (16) allows for locating new points, referred to as infill points, in promising areas, close to locations where the best fitness function value has been computed, as well as in less explored ones, characterized by a low sampling density and, consequently, high uncertainties.

Once the infill point is chosen as the sample maximizing the EI infill criterion, the expensive model (FEM) is there evaluated. If the value of the minimized objective function is smaller than the best so far, the best design is updated. In any case, the new data are used to update the Kriging approximation model. The infill procedure is repeated until a prescribed number of evaluations is reached.

4.2. Constraint Handling Techniques

During the DoE phase of the optimization algorithm, the sample points associated with infeasible designs, according either to the connectivity or to the volume constraint, are automatically discarded and only the feasible points are used to construct the surrogate model. Moreover, different techniques were developed within this research to deal with the connectivity and volume constraint during the infill procedure.

4.2.1. Expected Improvement for Connected Designs

The Expected Improvement for Connected Designs (EICD) is a variant of the standard EI, which was developed within this study to ensure the connectivity of the promising candidates during the infill procedure. The main idea is to penalize the infeasible designs according to the level of infeasibility. Thanks to a mapping to a two-dimensional graph representation (Figure 4), when disconnected designs are met during the maximization of EI by DE, the algorithm computes a penalty \( P \), which takes into account the amount of violation of the connectivity constraint:

\[
P = \gamma(P_1 + P_2 + P_3),
\]

(17)

where each \( P_i \), for \( i = 1, 2, 3 \), is the minimum extra-distance that has been computed according to the 1st, 2nd, or 3rd type of disconnection and \( \gamma \) is a suitable penalty factor. With reference to Figure 4, the implemented algorithm allows for some distances to be admitted in order to avoid a too strict connectivity criterion. In fact, the connectivity check is based on the graph representation, which is composed of nodes and edges. While an edge is a segment, beam components possess a certain thickness that makes the design feasible even if the corresponding graph representation is slightly not.
The penalty in Equation (17) is then used to modify the EI criterion (16) as follows:

$$
EICD(x) = \begin{cases} 
E[I(x)] & \text{if } x \text{ is connected}, \\
-P(x) & \text{if } x \text{ is disconnected}, 
\end{cases}
$$

(18)

where the introduction of the penalty $P$ is dictated by the necessity to avoid the creation of large plateaus in the optimization landscape, which might cause stagnation of the DE search. As a result, the designs infeasible according to the connectivity constraint are automatically discarded in the maximization of Equation (18). It is worth noting that the graph representation of a structure and the computation of the associated penalty in case if it does not satisfy the connectivity constraint represent a much cheaper procedure compared to the FEM evaluation regardless of the design feasibility. Therefore, since the convergence speed is measured in terms of evaluations, the proposed strategy enhances the convergence speed towards the optimum.

4.2.2. Constrained Expected Improvement

In this work, the Constrained Expected Improvement (CEI) [40] is used to drive the infill search towards designs satisfying a prescribed volume limit. At each iteration of the optimization strategy, when the Kriging surrogate is fitted on the previous training set and the EI function is computed, a second approximation is built for the function defining the volume constraint. If $Y(x) = N(\hat{y}(x), \hat{\sigma}^2(x))$ is the Kriging model for the objective function, the surrogate model for the constraint is:

$$
G(x) = N(\hat{g}(x), \hat{\sigma}^2(x)),
$$

(19)

where $\hat{g}(x)$ is the prediction of the constraint function and $\hat{\sigma}^2(x)$ is the variance of the model of the constraint. Hereafter, the Probability of Feasibility (PF), i.e. the probability that the design satisfies the volume constraint, is computed as:

$$
P[F(x)] = \frac{1}{\hat{\sigma}^2(x)\sqrt{2\pi}} \int_0^\infty \exp \left( -\frac{(F - \hat{y}(x))^2}{2\hat{\sigma}^2(x)} \right) dG,
$$

(20)
where $F = G(x) - V_{req}$ is the measure of feasibility. Hence, the probability that a new infill point improves on the best value so far and is feasible at the same time can be computed by maximizing the product between EI and PF:

$$x_{infill} = \arg \max_x \text{CEI}(x) = \arg \max_x E[I(x)]P[F(x)],$$

where the product is justified by the independence of the models.

In case if both of the constraints are taken into account, the coupling between EICD and CEI is here defined by applying the connectivity check in the maximization of EI-PF, leading to the following combined formulation:

$$EICD(x) = \begin{cases} \text{CEI}(x) & \text{if } x \text{ is connected} \\ -P(x) & \text{if } x \text{ is disconnected} \end{cases},$$

If the problem to be optimized is characterized by multiple inequality constraints, it is possible to extend the above derivation by considering the joint probability distribution of all the constraints when computing the PF of a design.

### 4.2.3. CMA-ES Baseline

In this work, the CMA-ES is chosen to validate and evaluate the potential of the proposed optimization method. Unlike the KG-LSM, it uses the exterior penalty method \cite{64} to guarantee the volume requirements in the alternative optimization algorithm, where the objective function is redefined by adding a term that produces a high cost whenever the constraint is violated:

$$f(x) = f_{obj}(x) + c \cdot \max(0, g(x)),$$

where $f_{obj}$ is the objective function minimized in the original problem and $c$ denotes an opportune penalty constant.

### 5. Test cases

In this research, the proposed surrogate-based strategy is first validated on the static cantilever beam problem and then applied to a dynamic transverse bending test case. Both of those test cases are presented in the following section.

#### 5.1. Linear elastic case

The considered static tests are performed on the standard cantilever beam benchmark problem, where the compliance of the structure is the objective to be minimized:

$$\min_x C(x)$$

subject to $g_1(x) = V(x) - V_{req} \leq 0, \quad x \in \mathbb{R}^k,$

where compliance $C$ is considered as the objective function and volume is the only active constraint, requiring the volume of the structure $V$ to be lower than or equal to the 50% of the design domain volume. In order to evaluate the potential and limits of the proposed strategy, the static optimization problem defined by Equation (24) is evaluated
for different dimensionalities ($k = 2, 9, 15$). Here, the beam is fixed to a support at the left-hand side and a static unit load is applied in the middle of the right-hand side (Figure 5a). The LSF is here mapped to a CalculiX FEM mesh, where a very low density space (1% of the material density) is assigned to the areas occupied by void. The domain dimensions are $20 \times 10$[mm] and it is discretized with 5000 four-node square shell (S4R) finite elements, arranged in a $100 \times 50$ grid. The test settings are shown in Table 1.

![Figure 5: Cantilever Beam test case: (a) problem definition and (b) CalculiX FEM mesh.](image)

<table>
<thead>
<tr>
<th>Property</th>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam material density</td>
<td>$\rho$</td>
<td>$7,85 \times 10^3$</td>
<td>kg/m$^3$</td>
</tr>
<tr>
<td>Young’s modulus</td>
<td>$E$</td>
<td>$2.1 \times 10^5$</td>
<td>MPa</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>$\nu$</td>
<td>0.3</td>
<td>-</td>
</tr>
<tr>
<td>Load</td>
<td>$F$</td>
<td>1</td>
<td>N</td>
</tr>
<tr>
<td>Required volume fraction</td>
<td>$V_f$</td>
<td>50%</td>
<td>-</td>
</tr>
<tr>
<td>Mesh resolution</td>
<td>-</td>
<td>100 x 50</td>
<td>-</td>
</tr>
<tr>
<td>Element type</td>
<td>-</td>
<td>Four-node shell element</td>
<td>-</td>
</tr>
<tr>
<td>Solver</td>
<td>-</td>
<td>CalculiX 2.9</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 1: Cantilever Beam test case settings.

In order to validate the presented method, two diagonal initial layouts of the local LSFs, as depicted in Figure 6, are used. The 2-beams design (Figure 6a) is taken as initial reference configuration to evaluate the KG-LSM performance in the optimization of mechanical structures with 2 free parameters: the $y$ coordinate of the barycenter and the orientation of each beam. The 6-beams reference layout (Figure 6b) is used instead to evaluate the method in a 9-variables environment first, where both the $x$ and $y$ coordinates of the beams barycenter as well as their orientation are free to vary during the optimization, and in a 15-variables test afterwards, where all the parameters defining the local LSFs are allowed to change. In each of the presented static cases, a symmetry of the design with respect to the domain horizontal symmetry axis is assumed throughout the optimization process.
Figure 6: Initial reference configurations for the static Cantilever Beam test case: (a) 2-beams case and (b) 6-beams case.

Figure 7: Transverse Bending test case: (a) problem definition and (b) LS-Dyna FEM mesh.

<table>
<thead>
<tr>
<th>Property</th>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam material density</td>
<td>( \rho )</td>
<td>( 2.7 \cdot 10^3 )</td>
<td>kg/m(^3)</td>
</tr>
<tr>
<td>Young’s modulus</td>
<td>( E )</td>
<td>( 7.0 \cdot 10^4 )</td>
<td>MPa</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>( \nu )</td>
<td>0.33</td>
<td>-</td>
</tr>
<tr>
<td>Yield strength</td>
<td>( R_y )</td>
<td>241.0</td>
<td>MPa</td>
</tr>
<tr>
<td>Tangent modulus</td>
<td>( E_{tan} )</td>
<td>70.0</td>
<td>MPa</td>
</tr>
<tr>
<td>Pole velocity</td>
<td>( v )</td>
<td>20</td>
<td>m/s</td>
</tr>
<tr>
<td>Pole mass</td>
<td>( m )</td>
<td>11.815</td>
<td>kg</td>
</tr>
<tr>
<td>Pole diameter</td>
<td>( D )</td>
<td>139.154</td>
<td>mm</td>
</tr>
<tr>
<td>LS-Dyna termination time</td>
<td>( t_{end} )</td>
<td>1.5</td>
<td>ms</td>
</tr>
<tr>
<td>LS-Dyna mesh resolution</td>
<td>-</td>
<td>80 x 20</td>
<td>-</td>
</tr>
<tr>
<td>Solver</td>
<td>-</td>
<td>LS-DYNA R7.1.1 [66]</td>
<td>-</td>
</tr>
<tr>
<td>Element type</td>
<td>-</td>
<td>Eight-node solid element</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 2: Configuration of the Transverse Bending test case.

5.2. Nonlinear crash case

The proposed application consists of a 6-beams and 15-variables optimization problem, starting from the reference layout shown in Figure 8. During the optimization...
process, all the parameters characterizing the moving morphable components are allowed to vary while respecting a symmetry condition with respect to the domain vertical symmetry axis.

For the dynamic test, the transverse bending case shown in Figure 7 is considered. The crash experiments involved a pole impact on a beam supported at both ends, as illustrated in Figure 7a. The optimization task is to minimize the intrusion of the impactor into the structure. This case is motivated by the standard pole impact crash test. In early phase design, structural criteria are considered; main criterion for pole impacts is the intrusion of the pole towards the driver’s head. In this paper, the intrusion is measured as maximal dynamic displacement of the rigid impactor before rebound.

Therefore, the optimization problem is defined as follows:

\[
\min_x \ d(x) \\
\text{s.t. } \ r(t) = 0, \quad g_1(x) = V(x) - V_{\text{req}} \leq 0, \quad x \in \mathbb{R}^{15},
\]

(25)

where \(d\) is the intrusion to be minimized, and the active constraint is still chosen as the volume of the structure \(V\) to be lower than or equal to the 50% of the design domain volume (similar tests can be performed with a comparable constraint on the mass of the structure). As given by Equation (25), the proposed application is tested for a dimensionality \(k = 15\).

The LSF function is mapped on a reference LS-Dyna mesh, as depicted in Figure 7b. It is composed of 1600 eight-node solid elements with a piecewise linear plasticity material and fixed in the \(z\) direction throughout the optimization procedure. In order to assure a physically correct crash behavior, the elements in the areas occupied by void are deleted from the mesh. The properties of the material are shown in Table 2.

6. Results

In this section, a comparison of the proposed KG-LSM topology optimization approach with the state-of-the-art CMA-ES is presented. In Section 6.1, the KG-LSM performance is first evaluated in a constrained static environment in order to assess its major advantages and drawbacks. Afterwards, Section 6.2 investigates an application of

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the proposed method to a crash scenario. In order to evaluate the convergence properties, 30 optimization runs of the considered methods are completed for each test case. The optimizations are stopped after 500 evaluations.

6.1. Linear elastic case

The considered methods are first evaluated on a static Cantilever Beam benchmark test case. The results obtained for the 2-variables, 9-variables, and 15-variables static test cases are presented below.

6.1.1. 2-variables static case

In view of an application to higher-dimensional optimization problems, a 2-variables test case is used in this work to evaluate the convergence behavior of KG-LSM when compared to CMA-ES. Without introducing any check on the connectivity and the volume constraints to be satisfied, the KG-LSM starting from a 20-samples DoE and the CMA-ES convergence trends towards the optimum are shown in Figure 9, where the current optima are recorded at each fitness function evaluation for both of the optimization methods and every value is normalized with respect to the compliance value of the reference design in Figure 6a. It is immediately visible that, while the curve corresponding to KG-LSM is monotonic, CMA-ES can return worse fitness values as the optimization moves on. This is due to the intrinsic nature of the surrogate-based algorithm and the evolution strategy. At each iteration, KG-LSM looks for a promising infill point, which is evaluated and substituted to the current optimum if it leads to a lower value of the objective function. Hence, the information about the current optimum is stored and plotted throughout the optimization procedure. CMA-ES works differently: at each iteration, the offspring is generated by means of a mutation and recombination process from the set of parent individuals and, since we are using a non-elitist approach, the best mutants become parents of the next generation, while the current parents are discarded. At this stage, the optimum is chosen as the best offspring individual. Therefore, after mutation and recombination, it can happen that the current optimum is worse than the one from the preceding generation, leading to the non-monotonic trend.

Moreover, on the one hand, it has to be noted that the solid line, representing the convergence behavior of the surrogate-based strategy, is characterized by a very steep decrease as soon as the DoE phase is over. Such improvement is due to the very good explorative abilities of the Expected Improvement infill criterion. On the other hand, the CMA-ES seems more suitable for exploiting the best design obtained so far and steadily converges to the minimum as the number of calls to the objective function increases, leading to a better performance over KG-LSM after evaluation 130. Although the ability of KG-LSM to rapidly converge towards an optimized structure can be considered sufficient to appreciate the potential of such strategy in optimization contexts where a fixed number of evaluations can be afforded, crashworthiness problems among all, it is worth wondering if improvements can arise by imposing structural constraints.

Therefore, KG-LSM starting from a DoE of 500 samples, where only the designs that satisfy the connectivity constraint are selected and evaluated, is also considered and renamed KG-LSM-EICD. Its convergence behavior is represented by the dashed line in Figure 9. Due to the simplicity of the studied test case, the designs connected to both the left-hand-side support and the right-hand-side load already result in very good objective
function values. This explains the much lower initial value of the fitness function and the small gap between the latter and the compliance characterizing the optimized structure. In any case, the selection of the connected designs as starting points of the optimization procedure and the EICD infill criterion produced a good and rapid convergence towards the same optimum found by CMA-ES. It has to be noted that there was no reason to consider the volume constraint in the 2-variables test case. In fact, the beams can only vary in orientation and $y$-position of the barycenter, making it impossible to overtake the prescribed volume limit allowed for the structure, which is set here to the 50% of the design domain.

When it comes to the optimized designs obtained with the considered strategies, they are all physically consistent and in accord with Mitchell’s theoretical model [67], as shown in Figure 10.

6.1.2. 9-variables static case

After the 2-variables case, the developed surrogate-based technique is tested on a 6-beams 9-variables configuration. Here, the orientation, the $x$ and $y$ positions of the beam barycenter go through the optimization process. Therefore, it is necessary to handle both the connectivity and the volume constraints, as described in Algorithm 1. Since CMA-ES converges towards the optimum by means of a recombination and mutation process, just the volume constraint is required to be fulfilled by using the exterior penalty method (as described in Section 4.2.3). In the discussion that follows, all the objective function values are normalized with respect to the reference design for the 6-beams test case defined in Figure 6b.
Figure 10: Optimized structure at evaluation 500 for CMA-ES (a), KG-LSM (b), and KG-LSM-ECID (c) compared to Mitchell theoretical model (d) for a 2-beams static test case.

Figure 11: Static 9-variables cantilever beam test case. Comparison of CMA-ES and KG-LSM starting from a 300-samples DoE, with selection of connected designs (amounting to an average value of 19 samples). Convergence of the compliance function averaged over 30 runs.

In Figure 11, a less evident gap between the starting points of the CMA-ES and KG-LSM plots can be observed. This is due to the fact that the target structures of this test case are more complex than the ones of the 2-variables static case, leading to a weaker link between that satisfaction of the constraints and the performance of the feasible structure. Anyway, similar considerations can be drawn. While the KG-LSM explorative capabilities lead to a fast convergence of the objective function towards the optimum at the beginning of the optimization procedure, CMA-ES steps over the convergence line of the surrogate-based optimization strategy around evaluation 250 due to its high explorative potential.
Although an evident gap is visible between the fitness function values of the optimum designs found by CMA-ES and KG-LSM, both of them converge to an optimal design that is consistent with the theoretical layout introduced by Mitchell. This correlation is shown in Figure 12.

![Figure 12: Best structures of 30 runs at evaluation 500 found by CMA-ES (a) and KG-LSM (b) compared to Mitchell theoretical model (c) for a 6-beams static test case.](image)

### 6.1.3. 15-variables static case

The same 6 beams configuration of Figure 6b is also used as reference design for a 15-variables static test case. Here, the dimensionality of the problem is obtained by letting all the beams parameters vary during the optimization process, while maintaining a symmetric design with respect to the $x-$axis throughout the optimization process. Since the length and the thickness of each component can vary during the optimization, it is essential to request that both of the connectivity and volume constraints are satisfied, according to Algorithm 1.

![Figure 13: Static 15-variables cantilever beam test case. Comparison of CMA-ES and KG-LSM starting from a 600-samples DoE, with selection of connected designs (amounting to an average value of 29 samples). Convergence of the compliance function averaged over 30 runs.](image)

Figure 13 presents the convergence trend of KG-LSM starting from a 600-samples
DoE, where only the connected designs are selected and evaluated, compared to CMA-ES. As in the previous test cases, all the fitness function values are normalized according to the 6-beams reference layout. It can be observed that, even if the dimensionality of the problem is increased, KG-LSM shows a better performance until evaluation 200, which is followed by a comparable convergence behavior for about 100 evaluations, leading to a preferable exploitive trend of the CMA-ES strategy at the end of the considered budget of evaluations.

The layouts resulting from each of the compared optimization strategies are not satisfactory, revealing that a higher number of evaluations is needed to converge towards a physically consistent structure as the dimensionality of the problems increases. Nevertheless, an application of the developed surrogate-based technique to a crash optimization scenario is investigated in the next section. Crashworthiness problems are commonly expensive-to-evaluate and, as such, optimization algorithms performing well in a limited number of available evaluations might represent good alternatives to the state-of-the-art optimizers. The KG-LSM application to a nonlinear crash case is presented in the next section.

6.2. Nonlinear crash case

A 6-beams 15-variables problem is here introduced to evaluate the KG-LSM performance in a constrained dynamic environment. A reference configuration of 6 beams arranged within the design domain according to Fig. 8 is considered. A symmetry of the design with respect to the domain vertical symmetry axis is assumed and all the beams parameters are allowed to change during the optimization process. Starting from a 200-samples DoE, the developed KG-LSM is firstly compared to CMA-ES. Afterwards, the DoE-CMA-ES method, which is initialized by choosing the first population of parent individuals as the best DoE sample available from the surrogate-based procedure, is also considered. In Figure 14, the Kriging-guided method convergence properties result to be superior to both the Evolution Strategies throughout the optimization process. Since KG-LSM outperforms DoE-CMA-ES, it can be deducted that the good convergence properties of the proposed optimization approach do not exclusively depend on the initial DoE procedure. Moreover, for each method, runs converging towards topologically equivalent layouts are monitored after 100, 250 and 500 evaluations. The analysis of such designs reveal that KG-LSM is able to obtain a structure with two connections to each support since evaluation 100 and gets close to a “two-crosses” layout at evaluation 250. On the contrary, structures that are far away from the sought design are found by both CMA-ES and DoE-CMA-ES at evaluation 100 as well as 250. In addition, the final topology reached by CMA-ES is of lower potential if compared to the layouts by the other methods at evaluation 500, pointing out the lower explorative capabilities of such evolution strategy.

In fact, as shown in Table 3, the third topology type, resulting in a two-beams structure, is the most commonly reached by both of the ESs, CMA-ES in particular. This fact stresses the tendency of CMA-ES to converge towards local optima due to its predominant exploitive skills. Instead, the optimal topology mostly derived by KG-LSM

\[\text{The visual inspection for the CMA-ES method is based on the final designs instead of the structures at evaluation 500 due to the fact that no topology classes could be distinguished at that stage of the optimization process.}\]
Figure 14: Comparison of the optimizations’ average convergence over 30 runs of the KG-LSM method, CMA-ES and DoE-CMA-ES for the 15-variables transverse bending test case. In KG-LSM and DoE-CMA-ES, the algorithm selects the DoE connected designs (amounting to an average value of 35 samples). Partial results for the optimization of topologically equivalent structures are shown after 100, 250 and 500 evaluations. The designs are ordered according to the legend and colors.

Table 3: Main topology types and frequencies with which they have been observed, ordered according to decreasing structural performance. Out of the 30 optimizations, the topology classification is made by visual inspection and it is based on the main characteristics determining the design (number of connections to the supports, final beam components, presence of cross sub-configurations).

<table>
<thead>
<tr>
<th>Topology type</th>
<th>KG-LSM</th>
<th>CMA-ES</th>
<th>DoE-CMA-ES</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3%</td>
<td>0%</td>
<td>3%</td>
</tr>
<tr>
<td></td>
<td>23%</td>
<td>7%</td>
<td>13%</td>
</tr>
<tr>
<td></td>
<td>17%</td>
<td>57%</td>
<td>23%</td>
</tr>
</tbody>
</table>

corresponds to the optima from the literature [34]. Figure 15 shows that, despite the low number of parameters, reasonably good structures can be obtained thanks to the used parametrization. By means of the proposed KG-LSM, the material distributions in Figure 15a, 15b and 15c which are ordered according to their performance, are found.
If compared to the first, second and third optimum available from the analysis of the crash test case in the work by Bujny et al. [34], the layouts in Figures 15a and 15b show many similarities. In particular, KG-LSM turned out to be able to find well-performing structures with a much lower computational effort than needed to reach the optimum by using Evolution Strategies. The discrepancies in the order with which the optima structures are presented in Figure 15 is due to the different dimensionalities of the problem considered here and the one from the literature [34]. When starting from a set of 6 beams, the structure in Figure 15a reaches a lower fitness value than the one in Figure 15b. On the contrary, the topologies in Figures (15d-15f) were obtained by using 16 moving morphable components and the greater amount of available beams lead to a more accurate “two-crosses” layout (Figure 15b), which becomes the most performing one.

The obtained results from both the convergence and the final designs point of view underline significant capabilities of the proposed surrogate-based method in providing useful indications for the practical construction of structural components and a sufficient reason to go for further investigations. However, it is worth noting that for a 15-variables problem, large differences in the optimized layouts resulted from the static and the crash load cases. This inconsistency might be motivated by the multi-modality characterizing the crash case, which makes the problem much more difficult for CMA-ES. As a result, the algorithm is often trapped in the local optima. On the other hand, KG-LSM performs more explorative search and is able to find much more frequently the global optimum or very good local optima.

In addition, the box plots in Figure 16 show the statistics of the considered strategies after 100, 250 and 500 evaluations and confirm the lower optimum values obtained by KG-LSM throughout the optimization process. In order to compare the mean values reached by the fitness function at evaluation 500 for each optimization method, the statistical Wilcoxon rank sum test was performed. The null hypothesis that data from the different optimization methods have equal medians at the 5% significance level was rejected when comparing both the ESs with KG-LSM, supporting that the KG-LSM leads to solutions with better performance than the other tested approaches.

Finally, the performance of the structures obtained with KG-LSM is compared with the ones obtained with state-of-the-art methods for crash topology optimization.

As the first approach, a simple version of the Equivalent Static Loads (ESL) method
is used \[60\]. Here, the dynamic crash load is replaced with a single static force applied at the point of contact between the impactor and the structure. The material model is changed to the linear elastic with Young modulus $E = 7.0 \times 10^4$ and Poisson ratio $\nu = 0.33$, as well. After this simplification, the topology is optimized for minimal compliance and 50% volume constraint with use of a gradient-based SIMP approach \[68\]. The sensitivity filtering with filter size of 15 [mm] and a material penalization with power $p = 3$ is used. In order to obtain the 0-1 material distribution, the elements with densities below a given threshold $\rho_{th}$ are removed, while the remaining ones are assigned full density. The threshold $\rho_{th}$ is chosen in such a way that the resulting design satisfies the 50% volume constraint. Then, the resulting topology is used in a crash simulation with a setup as specified in Table 2.

The second state-of-the-art approach used for comparison with KG-LSM is the Hybrid Cellular Automata (HCA) method \[69\]. The method uses a heuristic energy homogenization criterion. The crash case as described in Section 5.2 is optimized with use of
LS-TaSC 3.2 with the default settings. As in case of KG-LSM and ESL, 50% volume constraint is imposed. The elements with intermediate densities are either deleted or assigned full densities using the approach described above, for the ESL approach. At the end, the performance of the post-processed topology is evaluated in a crash simulation.

Crash performance of the topologies obtained with KG-LSM, ESL and HCA is compared in Table 4. According to the values measured for intrusion, KG-LSM turns out to provide the best performing structure, although the small differences between the presented values indicate that all performances are comparable. It is worth noting that both ESL and HCA usually yield very good results for intrusion minimization problems [28, 33, 70]. What is more, the design freedom in KG-LSM is strongly restricted due to the low number of components used in the representation. As a result, the number of attainable topologies in KG-LSM is much smaller than both in ESL and HCA, which operate on the densities of single elements. Therefore, it is not surprising that the performance of the topologies obtained with KG-LSM, ESL and HCA is similar for the considered problem. Since KG-LSM is an approach free from heuristic assumptions and addressing the optimization problem directly, it should yield significantly better results than ESL and HCA for different types of problems, e.g. involving complex plastic behavior or injury-related optimization criteria.

<table>
<thead>
<tr>
<th>Performance comparison with state-of-the-art methods</th>
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</thead>
<tbody>
<tr>
<td>KG-LSM</td>
</tr>
<tr>
<td>Optimal layout</td>
</tr>
<tr>
<td>Intrusion value</td>
</tr>
</tbody>
</table>

Table 4: Comparison of crash performance of optimal topologies obtained with KG-LSM and with state-of-the-art methods (ESL and HCA).

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4http://www.lstc.com/products/ls-tasc
7. Conclusions

In this paper, the novel Kriging-Guided Level Set Method (KG-LSM) was presented and used in the optimization of structural topologies in both linear elastic and nonlinear crash test cases.

The method profits from an implicit parametrization introduced in precedent works [33, 37], which defines a set of local Level Set Functions representing several beam components allowed to move in the design domain within the optimization process, leading to a significant reduction of the amount of design variables commonly needed in structural Topology Optimization (TO). The state-of-the-art methods for crashworthiness optimization present some limitations due to the strong nonlinearities and discontinuities characterizing the problem. Therefore, the Kriging surrogate model was adopted in this work to drive the update process of the level set-based optimization. Since surrogate models do not need any sensitivity information but construct approximations by directly evaluating the real objective function, optimal structures were found without introducing any heuristic assumptions. Moreover, the Expected Improvement for Connected Designs (EICD) was developed and the Constrained Expected Improvement (CEI) adapted to deal with the constraints required by the considered structural problems. Their combination contributed to the good performance of the optimization algorithm. This is due to the fact that, unlike the standard Expected Improvement, the EICD, coupled with CEI, does not waste precious evaluations on the infeasible areas of the domain space, which would produce physically inconsistent structural layouts, and, hence, bad fitness function values.

The method was validated on a static cantilever beam test case, by comparing the convergence behavior of the proposed KG-LSM and the standard Covariance Matrix Adaptation Evolution Strategy (CMA-ES) in optimization problems of different dimensionalities. Afterwards, an application to a dynamic transverse bending test case was presented to show the suitability of the defined technique to situations commonly faced in structural mechanics.

The results from the static tests show that the proposed approach can be a valid alternative to evolutionary-based methods in the optimization of mechanical structures. Indeed, in the initial phase of the optimization process, significant improvements of the convergence properties could be observed for low to moderate search dimensionalities, if compared with the CMA-ES. Although the adopted parametrization allows for testing the developed KG-LSM in a restrained-dimensionality environment and hence for obtaining reasonable structures with a low number of variables, future effort is aimed to be focused to address higher-dimensional problems. In fact, high-dimensional surrogate-based optimization has been already treated in the literature. Some examples are embodied by the usage of principal components analysis (PCA) to guide the application of evolution operators and the training of the metamodel [71], the construction of a covariance kernel depending on only a few of the current Kriging model parameters [72], or the combination of the Kriging surrogate and the partial least squares method generating the KPLS(+K) model [73].

In any case, since in many industrial contexts and especially for expensive crash optimization problems a limited budget of evaluations is commonly available, a faster optimization process can be more favorable than determining a more precise mathematical optimum. Therefore, the fast convergence obtained at the beginning of the optimization
demonstrates to be a great advantage. Furthermore, since Evolution Strategies turned out to have great exploitive characteristics, hybrid techniques, which couple surrogates with evolutionary algorithms, are currently under investigation.

For the pole intrusion minimization problem, fast convergence capabilities and a good performance of the obtained designs were found, confirming the potential of the proposed surrogate-based strategy in the context of crash topology optimization and the chance to deal with other structural TO problems in many industrial applications, thanks to its generality.

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References


